

THE REGULARITY PROBLEM FOR SECOND ORDER ELLIPTIC OPERATORS WITH COMPLEX-VALUED BOUNDED MEASURABLE COEFFICIENTS

STEVE HOFMANN, CARLOS KENIG, SVITLANA MAYBORODA, AND JILL PIPHER

ABSTRACT. The present paper establishes a certain duality between the Dirichlet and Regularity problems for elliptic operators with t -independent complex bounded measurable coefficients (t being the transversal direction to the boundary). To be precise, we show that the Dirichlet boundary value problem is solvable in $L^{p'}$, subject to the square function and non-tangential maximal function estimates, if and only if the corresponding Regularity problem is solvable in L^p . Moreover, the solutions admit layer potential representations.

In particular, we prove that for any elliptic operator with t -independent real (possibly non-symmetric) coefficients there exists a $p > 1$ such that the Regularity problem is well-posed in L^p .

1. INTRODUCTION

We consider a divergence form elliptic operator

$$L := -\operatorname{div} A(x)\nabla,$$

defined in $\mathbb{R}^{n+1} = \{(x, t), x \in \mathbb{R}^n, t > 0\}$. Here A is an $(n+1) \times (n+1)$ matrix of bounded, complex-valued, t -independent coefficients, which satisfies the uniform ellipticity condition

$$(1.1) \quad \lambda|\xi|^2 \leq \operatorname{Re} \langle A(x)\xi, \xi \rangle := \operatorname{Re} \sum_{i,j=1}^{n+1} A_{ij}(x) \xi_j \bar{\xi}_i, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq \lambda^{-1},$$

for some $\lambda > 0$, and for all $\xi \in \mathbb{C}^{n+1}$, $x \in \mathbb{R}^n$. The number λ in (1.1) will be referred as the ellipticity parameter of L . As usual, the divergence form equation is interpreted in the weak sense, i.e., we say that $Lu = 0$ in a domain Ω if $u \in W_{loc}^{1,2}(\Omega)$ and

$$(1.2) \quad \iint_{\Omega} A \nabla u \cdot \overline{\nabla \Psi} = 0,$$

for all complex valued $\Psi \in C_0^\infty(\Omega)$. For us, Ω will be a Lipschitz graph domain

$$(1.3) \quad \Omega_\psi := \{(x, t) \in \mathbb{R}^{n+1} : t > \psi(x)\},$$

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where $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a Lipschitz function, or more specifically (but without loss of generality), Ω will be the half-space $\mathbb{R}_+^{n+1} := \{(x, t) \in \mathbb{R}^n \times (0, \infty)\}$. We shall return to this point below.

We say that the Dirichlet problem $(D_{p'})$ for L is well-posed for some $1 < p' < \infty$ if given any $f \in L^{p'}(\mathbb{R}^n)$ there exists a unique solution to the boundary problem

$$(D_{p'}) \quad \begin{cases} Lu = 0 \text{ in } \mathbb{R}_+^{n+1} \\ \lim_{t \rightarrow 0} u = f \text{ n.t.} \\ \|N_*(u)\|_{L^{p'}(\mathbb{R}^n)} < \infty, \end{cases}$$

and

$$(1.4) \quad \|N_*(u)\|_{L^{p'}(\mathbb{R}^n)} \leq C\|f\|_{L^{p'}(\mathbb{R}^n)}.$$

Here, the notation “ $\lim_{t \rightarrow 0} u = f$ n.t.” means that $\lim_{(y,t) \rightarrow (x,0)} u(y,t) = f(x)$, for a.e. $x \in \mathbb{R}^n$, where the limit runs over $(y,t) \in \Gamma(x) := \{(y,t) \in \mathbb{R}_+^{n+1} : |y-x| < t\}$, and

$$N_*F(x) \equiv \sup_{(z,t) \in \Gamma(x)} |F(z,t)|.$$

The Regularity problem (R_p) for L is well-posed for some $1 < p < \infty$ if given any $f \in \dot{L}_1^p(\mathbb{R}^n)$ there exists a unique solution to the boundary problem

$$(R_p) \quad \begin{cases} Lu = 0 \text{ in } \mathbb{R}_+^{n+1} \\ \lim_{t \rightarrow 0} u = f \text{ n.t.} \\ \|\tilde{N}(\nabla u)\|_{L^p(\mathbb{R}^n)} < \infty, \end{cases}$$

and

$$(1.5) \quad \|\tilde{N}(\nabla u)\|_{L^p(\mathbb{R}^n)} \leq C\|\nabla_\parallel f\|_{L^p(\mathbb{R}^n)}.$$

The homogeneous Sobolev space $\dot{L}_1^p(\mathbb{R}^n)$ is the completion of C_0^∞ with respect to the Sobolev norm $\|\nabla_\parallel f\|_{L^p(\mathbb{R}^n)}$. While fairly evident here, it will be convenient to distinguish the gradient in \mathbb{R}^{n+1} and the gradient in \mathbb{R}^n throughout the paper, and we shall denote the latter by ∇_\parallel . The modified non-tangential maximal function is given by

$$\tilde{N}F(x) \equiv \sup_{(z,t) \in \Gamma(x)} \left(\iint_{W(z,t)} |F(y,s)|^2 dy ds \right)^{\frac{1}{2}},$$

where $W(x,t) \equiv \Delta(x,t) \times (t/2, 3t/2)$, $\Delta(x,t) \equiv \{y \in \mathbb{R}^n : |x-y| < t\}$.

We note, furthermore, that $(D_{p'})$ and (R_p) above are defined in \mathbb{R}_+^{n+1} . An analogous definition applies to the lower half-space. Well-posedness in \mathbb{R}_\pm^{n+1} stands for the well posedness both in \mathbb{R}_+^{n+1} and \mathbb{R}_-^{n+1} , and similarly for other properties.

It has been proved in [HKMP] that for any elliptic operator L with real bounded measurable t -independent coefficients there exists $p' < \infty$ such that the Dirichlet problem $(D_{p'})$ is well-posed. The purpose of this paper is to establish a certain duality between the Dirichlet and the Regularity problems, and in particular, to show that for any elliptic operator L with real bounded measurable t -independent coefficients there exists a $p > 1$ such that the Regularity problem (R_p) is well-posed. Before stating the main result, let us introduce some relevant terminology.

Here and throughout the paper, the capital letters X, Y, Z denote points in \mathbb{R}^{n+1} and the corresponding small ones stand for the points in \mathbb{R}^n . Furthermore, $B = B_R(X) = B(X, R)$ is the ball in \mathbb{R}^{n+1} centered at $X \in \mathbb{R}^{n+1}$ with the radius $R > 0$, and $\Delta = \Delta_R(x) = \Delta(x, R)$ is the ball in \mathbb{R}^n centered at $x \in \mathbb{R}^n$ with the radius $R > 0$. Then the tent regions are $T(\Delta) = \widehat{\Delta} = \{(x, t) \in \mathbb{R}_+^{n+1} : \text{dist}(x, \Delta^c) \geq t\}$, and the Whitney cubes are, as above, $W(x, t) = \Delta(x, t) \times (t/2, 3t/2)$, $(x, t) \in \mathbb{R}_+^{n+1}$.

Throughout the paper L will be an elliptic divergence form elliptic operator with bounded, measurable, complex-valued, t -independent coefficients. We shall assume, in addition, that the solutions to $Lu = 0$ in \mathbb{R}_+^{n+1} are locally Hölder continuous in the following sense. Assume that $Lu = 0$ in \mathbb{R}_+^{n+1} in the weak sense and $B_{2R}(X) \subset \mathbb{R}_+^{n+1}$, $X \in \mathbb{R}_+^{n+1}$, $R > 0$. Then

$$(1.6) \quad |u(Y) - u(Z)| \leq C \left(\frac{|Y - Z|}{R} \right)^\mu \left(\iint_{B_{2R}(X)} |u|^2 \right)^{\frac{1}{2}}, \quad \text{for all } Y, Z \in B_R(X),$$

for some constants $\mu > 0$ and $C > 0$. In particular, one can show that for any $p > 0$

$$(1.7) \quad |u(Y)| \leq C \left(\iint_{B_{2R}(X)} |u|^p \right)^{\frac{1}{p}}, \quad \text{for all } Y, Z \in B_R(X).$$

We shall refer to property (1.6) by saying that the solutions (or, slightly abusing the terminology, the operator) satisfy the De Giorgi-Nash-Moser bounds. Respectively, the constants C and μ in (1.6), (1.7) will be referred to as the De Giorgi-Nash-Moser constants of L . Finally, following [AAAHK], [HMiMo], [HMM], we shall normally refer to the following collection of quantities: the dimension, the ellipticity, and the De Giorgi-Nash-Moser constants of L , L^* collectively as the “*standard constants*”.

We note that the De Giorgi-Nash-Moser bounds are not necessarily satisfied for all elliptic PDEs with complex t -independent coefficients [F, MNP, HMMc]. However, (1.6), (1.7) always hold when the coefficients of the underlying equation are real [DeG, Na, Mo], and the constants depend quantitatively only upon ellipticity and dimension (for this result, the matrix A need not be t -independent). Moreover, (1.6) (which implies (1.7)) is stable under small complex perturbations of the coefficients in the L^∞ norm (see, e.g., [Gi], Chapter VI, or [A1]). Thus, in particular, (1.6)-(1.7) hold automatically, e.g., for small complex perturbations of real elliptic coefficients. We also note that in the t -independent setting that we consider here, the interior the De Giorgi-Nash-Moser/M estimates hold always when the ambient dimension $n + 1 = 3$ (see [AAAHK, Section 11]).

The solutions to elliptic operators with *real* coefficients (and their complex perturbations) also satisfy the *boundary Hölder continuity condition*:

Assume that $Lu = 0$ in $T(\Delta_{2R}(x))$, $x \in \mathbb{R}^n$, $R > 0$, and $u \equiv 0$ on $\Delta_{2R}(x)$. Then

$$(1.8) \quad |u(y, s)| \leq C \left(\frac{s}{R} \right)^\mu \left(\iint_{T(\Delta_{2R}(x))} |u|^2 \right)^{\frac{1}{2}}, \quad \text{for all } (y, s) \in T(\Delta_R(x)),$$

for some $\mu > 0$ and $C > 0$ depending on the dimension and the ellipticity parameter of L only. It is not clear, however, whether the interior Hölder continuity implies boundary Hölder continuity for a general elliptic operator with bounded measurable complex coefficients.

Let us now turn to the layer potentials. Let L be an elliptic operator with bounded measurable coefficients. By E , E^* we denote the fundamental solutions associated with L and L^* , respectively, in \mathbb{R}^{n+1} , so that

$$L_{x,t} E(x, t; y, s) = \delta_{(y,s)}(x, t) \quad \text{and} \quad L_{y,s}^* E^*(y, s; x, t) \equiv L_{y,s}^* \overline{E(x, t; y, s)} = \delta_{(x,t)}(y, s),$$

where $\delta_{(x,t)}$ denotes the Dirac delta function at the point (x, t) . One can refer, e.g., to [HK] for their construction and properties. We note for future reference that when the coefficients of the underlying matrix are t -independent,

$$(1.9) \quad E(x, t; y, s) = E(x, t - s; y, 0),$$

and hence, in particular, one can swap the derivatives in t and s for the fundamental solution.

The single layer potential and the double layer potential operators associated with L are given, respectively, by

$$(1.10) \quad S_t^L f(x) \equiv \int_{\mathbb{R}^n} E(x, t; y, 0) f(y) dy, \quad t \in \mathbb{R}, x \in \mathbb{R}^n,$$

$$D_t^L f(x) \equiv \int_{\mathbb{R}^n} \overline{\partial_{\nu_A^*, y} E^*(y, 0; x, t)} f(y) dy, \quad t \neq 0, x \in \mathbb{R}^n.$$

Here, the conormal derivative is roughly $\partial_{\nu_A} u = -e_{n+1} A(y) \nabla u$, $e_{n+1} = (0, \dots, 0, 1)$. The precise meaning of the latter on the boundary will be discussed later, with the Preliminaries.

Going further, it will be convenient to separate the issue of *well-posedness* of $(D_{p'})$, (R_p) as defined above, from *solvability*. We shall say that $(D_{p'})$ (resp., (R_p)) is *solvable* if for every $f \in C_0^\infty(\mathbb{R}^n)$ there exists some function $u \in W_{loc}^{1,2}(\mathbb{R}_+^{n+1})$ such that $(D_{p'})$ (resp., (R_p)) is valid and (1.4) (resp., (1.5)) holds. Throughout the manuscript, the constants in (1.4), (1.5) will be referred to as the solvability constants of L . We say, furthermore, that $(D_{p'})$ (resp., (R_p)) is *well-posed with the Green formula representation* if for every $f \in L^{p'}$ (resp., $f \in \dot{L}_1^p$) there exists a *unique* $u \in W_{loc}^{1,2}(\mathbb{R}_+^{n+1})$ such that $(D_{p'})$ (resp., (R_p)) is valid, (1.4) (resp., (1.5)) holds, and

$$u = -D_t^L(f) + S_t^L(\partial_{\nu_A} u).$$

For reasonably nice solutions the formula above is simply the result of integration by parts. However, in the context of the general elliptic operators with complex coefficients, it seems to be necessary to require its validity explicitly in order to establish uniqueness. See Section 7 for a detailed discussion.

The main result of this paper is as follows.

Theorem 1.11. *Let L be a divergence form elliptic operator with bounded, complex-valued, t -independent coefficients. Assume, in addition, that the solutions to $Lu = 0$ in \mathbb{R}_\pm^{n+1} satisfy the De Giorgi-Nash-Moser bounds, and that the same is true for the*

adjoint operator L^* . Then there exists $\varepsilon > 0$ depending on the standard constants only such that for any $1 < p < 2 + \varepsilon$, $\frac{1}{p} + \frac{1}{p'} = 1$, the following are equivalent:

- (a) The Dirichlet problem $(D_{p'})$ for L^* is solvable in \mathbb{R}_\pm^{n+1} , and, in addition to (1.4), the solution satisfies the square function bounds

$$(1.12) \quad \|\mathcal{A}(t\nabla u)\|_{L^{p'}(\mathbb{R}^n)} \leq C\|f\|_{L^{p'}(\mathbb{R}^n)},$$

where \mathcal{A} stands for the square function, that is,

$$(1.13) \quad \mathcal{A}F(x) := \left(\iint_{|x-y|<|t|} |F(y,t)|^2 \frac{dydt}{|t|^{n+1}} \right)^{1/2},$$

for $F : \mathbb{R}_\pm^{n+1} \rightarrow \mathbb{R}$.

- (b) The Regularity problem (R_p) for L is solvable in \mathbb{R}_\pm^{n+1} .
(c) The Regularity problem (R_p) for L is solvable in \mathbb{R}_\pm^{n+1} , and the solution can be represented by means of layer potentials, that is,

$$(1.14) \quad u(x, t) = S_t^L (S_0^L)^{-1} f(x), \quad (x, t) \in \mathbb{R}_\pm^{n+1}.$$

In particular, the operator $f \mapsto \tilde{N}(\nabla S_t^L f)$ is bounded in $L^p(\mathbb{R}^n)$ and

$$S_0^L : L^p(\mathbb{R}^n) \rightarrow \dot{L}_1^p(\mathbb{R}^n)$$

is invertible.

- (d) The Dirichlet problem $(D_{p'})$ for L^* is solvable in \mathbb{R}_\pm^{n+1} , and the solution can be represented by means of layer potentials as

$$(1.15) \quad u(x, t) = S_t^{L^*} (S_0^{L^*})^{-1} f(x), \quad (x, t) \in \mathbb{R}_\pm^{n+1}.$$

In particular, $f \mapsto N(S_t^{L^*} f)$ is a bounded operator from $L_{-1}^{p'}(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ and

$$S_0^L : L_{-1}^{p'}(\mathbb{R}^n) \rightarrow L^{p'}(\mathbb{R}^n)$$

is invertible. Here $L_{-1}^{p'}(\mathbb{R}^n) := (\dot{L}_1^p(\mathbb{R}^n))^*$.

The solutions to $(D_{p'})$ and (R_p) in (a) – (d) are unique in the class of solutions with the Green formula representation and, respectively, the corresponding boundary formulas are well-posed with the Green formula representation.

In combination with the results in [HKMP], [AAAHK], [HMiMo], [HMM], and [R] (as regards the latter, see also [GH] for an alternative proof), Theorem 1.11 yields the following Corollaries. Their proofs can be found in Section 8.

Corollary 1.16. *Let L be a divergence form elliptic operator with bounded measurable t -independent coefficients, and assume that there exists an operator L_0 , falling under the scope of Theorem 1.11, such that for some $p > 1$ one of the properties (a) – (d) is satisfied for L_0 . Assume, furthermore, that*

$$\|A - A_0\|_{L^\infty(\mathbb{R}^n)} < \eta$$

for a sufficiently small $\eta > 0$ depending on the standard constants and the involved solvability constants of L_0 , L_0^* . Then all four assertions (a) – (d) of Theorem 1.11 are valid for the operator L as well.

Corollary 1.17. *Let L be a divergence form elliptic operator with bounded measurable coefficients, and assume that there exists an operator L_0 , falling under the scope of Theorem 1.11, such that for some $p > 1$ one of the properties (a) – (d) is satisfied for L_0 . Denote*

$$\epsilon(x, t) := \sup \{|A(y, s) - A_0(y)| : (y, s) \in W(x, t)\}$$

where $W(x, t) \equiv \Delta(x, t) \times (t/2, 3t/2)$ and $\Delta(x, t) \equiv \{y \in \mathbb{R}^n : |x - y| < t\}$. Suppose further that ϵ satisfies the Carleson-type condition

$$(1.18) \quad \sup_{\text{cubes } Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|} \iint_{R_Q} \epsilon^2(x, t) \frac{dx dt}{t} \right)^{\frac{1}{2}} < \eta,$$

for a sufficiently small $\eta > 0$ depending on the standard constants and the involved solvability constants of L_0, L_0^* . Here $R_Q := Q \times (0, \ell(Q))$ is a Carleson box and $\ell(Q)$ is the side-length of the cube $Q \subset \mathbb{R}^n$.

Then all four assertions (a) – (d) of Theorem 1.11 are valid for the operator L as well.

Corollary 1.19. *Let L be either a divergence form elliptic operator with real (possibly non-symmetric) bounded t -independent coefficients, or a perturbation of such an operator, in the sense of Corollary 1.16 or in the sense of Corollary 1.17. Then there exists $p > 1$, depending on the dimension and the ellipticity constant of L only, such that statements (a) – (d) of Theorem 1.11 are valid for L . Moreover, the corresponding solutions to the Dirichlet and Regularity problems are unique.*

In particular, the Regularity problem is well-posed, for some range of $p > 1$, for any elliptic operator with real t -independent coefficients and for its perturbations.

We note that this result is sharp, in the sense that one cannot specify the range of well-posedness of boundary value problems for real non-symmetric t -independent operators which would not depend on the ellipticity parameter of the operator. More precisely, for every $p > 1$ there exists an elliptic operator L with real non-symmetric t -independent coefficients such that the Regularity problem (R_p) is not well-posed. A similar statement holds for the Dirichlet problem: given any $p' < \infty$ there exists an elliptic operator L with real non-symmetric t -independent coefficients such that the Dirichlet problem ($D_{p'}$) is not well-posed. The counterexample can be found in [KKPT] and [KR] in the context of the Dirichlet and Regularity problem, respectively.

In the case of real coefficients, the solvability of the Regularity problem for some L^p , $p > 1$, is equivalent to the solvability in Hardy spaces, much as the solvability of the Dirichlet problem in $L^{p'}$ for some $p' < \infty$ is equivalent to the solvability of the Dirichlet problem in BMO . For the Dirichlet problem the equivalence was established in [DKP] and for the Regularity problem in [DK]. We refer the reader to [DKP], [DK] for precise statements. Here we just point out that in the realm of real coefficients the results of Corollary 1.19 automatically extend to Hardy and BMO spaces for Regularity and Dirichlet problems, respectively.

Finally, we remark that all the results, in particular, Theorem 1.11 and Corollaries 1.16–1.19, automatically extend to Lipschitz domains as defined in (1.3).

This is a consequence of the fact that a “flattening” change of variables $(x, t) \mapsto (x, t - \psi(x))$, which maps a Lipschitz domain $\{(x, t) \in \mathbb{R}_+^{n+1} : t > \psi(x)\}$ into \mathbb{R}_+^{n+1} , preserves the class of t -independent elliptic operators.

Let us now discuss the history of the problem. The study of elliptic boundary problems $(D_{p'})$, (R_p) has started with the results for the Dirichlet problem for the Laplacian on Lipschitz domains [D1], [D2]. The first breakthrough in the context of the elliptic operators with bounded measurable coefficients came in [JK], where the authors realized how to resolve $(D_{p'})$, $p' = 2$, resting on the so-called Rellich identity. The latter is essentially a result of an integration by parts argument which allows one to compare the tangential and normal derivatives of the solution on the boundary, in the sense that

$$(1.20) \quad \|\nabla f\|_{L^2(\mathbb{R}^n)} \approx \|\partial_{\nu_A} u\|_{L^2(\mathbb{R}^n)}, \quad f = u|_{\mathbb{R}^n}.$$

The Rellich identity underpinned the development of the elliptic theory for real and symmetric operators, and over the years the problems $(D_{p'})$ and (R_p) were resolved for the sharp range of p for operators with real symmetric t -independent coefficients in [KP] and perturbation results in the spirit of Corollary 1.17 were obtained in [D3], [FJK], [FKP], [KP2], (see also [AAAHK], [AA], [HMM] for later developments in connection with the perturbation questions).

We remark that some “smoothness” of the underlying matrix in t is necessary for well-posedness [CFK] and thus starting the investigation with the t -independent case is natural in this context.

The argument for the Rellich identity heavily used the condition of the symmetry of the matrix, and thus, could not be extended neither to real non-symmetric, nor more generally, to the complex case. The only exception to this rule was the resolution of the Kato problem [CMcM, HMc, HLMc, AHLMcT], in which (1.20) was established in the absence of self-adjointness, in the special case that the matrix has block structure, that is, $A = \{A_{jk}\}_{j,k=1}^{n+1}$ with $A_{j,n+1} = A_{n+1,j} = 0$, $j = 1, \dots, n$. The observation that the solution of the Kato problem amounts to (1.20) is due to C. Kenig, see [K]. Moreover, quite recently (simultaneously with the preparation of this manuscript) it was shown in [AMM], by a refinement of the proof of the Kato conjecture, that the Kato estimate, or more precisely, the \gtrsim side of (1.20), could be extended to the “block triangular” case when only $A_{j,n+1} = 0$ without necessarily $A_{n+1,j} = 0$ and similarly, the \lesssim side of (1.20) holds when $A_{n+1,j} = 0$.

However, the aforementioned ideas could not be directly applied to the general case of a non-symmetric matrix lacking any additional block structure¹. Moreover, it was demonstrated in [KKPT] (see also [KR]) that the well-posedness in L^2 may fail when matrix has no symmetry and thus, (1.20) is not to be expected. Nonetheless, using a completely different approach, in [KKPT], [HKMP] the authors have established that for any operator L with real non-symmetric coefficients there is a $p' < \infty$ such that $(D_{p'})$ is solvable. This raised the question of solvability of the Regularity problem.

¹ although the technology of the Kato problem continues to play a crucial role in the present paper and in [HKMP].

In the particular case of the operators with real non-symmetric coefficients in dimension two and their perturbations the Regularity problem was resolved in [KR], [B]. The present paper resolves this problem in arbitrary dimension. It shows that the solvability of the Dirichlet problem is generally equivalent to that of the Regularity problem and thus, in the context of real non-symmetric matrices there is always a p such that (R_p) is solvable. The core of our argument is a *new Rellich-type inequality*. We demonstrate that, in fact, for any operator L with complex and t -independent coefficients the solvability of the Dirichlet problem $(D_{p'})$, together with the square function bounds, entails a one-sided Rellich inequality,

$$(1.21) \quad \|\partial_{\nu_A} u\|_{L^p(\mathbb{R}^n)} \lesssim \|\nabla_{\parallel} f\|_{L^p(\mathbb{R}^n)}, \quad f = u|_{\mathbb{R}^n}.$$

This ultimately paves the way to (R_p) . Of course, having a reverse inequality as well would be extremely interesting, but at the moment seems quite challenging.

Finally, we also point out that the actual question of connections between the Dirichlet and Regularity problem has received considerable attention in the literature, and some partial results were established in [V] $((R_p) \iff (D_{p'}))$, Laplacian on a Lipschitz domain), [KP] $((R_p) \implies (D_{p'}))$, real coefficients), [S], [KiS] $((R_p) \iff (D_{p'}))$, real symmetric constant coefficient systems), [KR] $((D_{p'}) \implies (R_p))$, real coefficients, dimension two). Some related counterexamples were obtained in [M]. Our main result, Theorem 1.11, generalizes all implications above, at least as far as the t -independent matrices are concerned. The proof of (1.21) builds on Verchota's duality argument [V], reducing matters to proving $L^{p'}$ estimates for certain conjugates, and in turn, the estimates for the conjugates will be obtained by an extension of an argument in [AAAHK] which exploits the solution of the Kato problem.

2. PRELIMINARIES

Let L be a divergence-form elliptic operator with t -independent bounded measurable coefficients. Any solution to $Lu = 0$ satisfies the *interior Caccioppoli inequality*:

Assume that $Lu = 0$ in \mathbb{R}_+^{n+1} in the weak sense and $B_{2R}(X) \subset \mathbb{R}_+^{n+1}$, $X \in \mathbb{R}_+^{n+1}$, $R > 0$. Then

$$(2.1) \quad \iint_{B_R(X)} |\nabla u(Y)|^2 dY \leq \frac{C}{R^2} \iint_{B_{2R}(X)} |u(Y)|^2 dY,$$

for some $C > 0$ depending on the dimension and the ellipticity parameter of L only. An analogous statement holds in \mathbb{R}_-^{n+1} .

Recall that we assume, in addition, that solutions of L and L^* satisfy the interior Hölder continuity conditions, that is, the De Giorgi-Nash-Moser estimates (1.6), (1.7).

We proceed to the issues of the non-tangential convergence and uniqueness of solutions. Unless explicitly stated otherwise, we assume throughout the rest of the paper that $L = -\operatorname{div}(A\nabla)$ is an elliptic operators with complex bounded measurable t -independent coefficients and that the solutions to $Lu = 0$ and $L^*u = 0$ satisfy the De Giorgi-Nash-Moser bounds. Furthermore, throughout the paper we assume that

$n \geq 2$, as much of this theory in the case $n = 1$ has already been treated in [KR] and [B].

Lemma 2.2. *Suppose that $u \in W_{loc}^{1,2}(\mathbb{R}_+^{n+1})$ is a weak solution of $Lu = 0$, which satisfies $\tilde{N}(\nabla u) \in L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. Then*

(i) *there exists $f \in \dot{L}_1^p(\mathbb{R}^n)$ such that $u \rightarrow f$ n.t. a.e., with*

$$|u(y, t) - f(x)| \lesssim t \tilde{N}(\nabla u)(x), \quad \text{for every } (y, t) \in \Gamma(x), x \in \mathbb{R}^n,$$

and

$$\|f\|_{\dot{L}_1^p} \lesssim \|\tilde{N}(\nabla u)\|_{L^p};$$

(ii) *for the limiting function f from (i), one has*

$$\nabla_{\parallel} u(\cdot, t) \longrightarrow \nabla_{\parallel} f$$

as $t \rightarrow 0$, in the weak sense in L^p ;

(iii) *there exists $g \in L^p(\mathbb{R}^n)$ such that $g = \partial_{\nu_A} u$ in the variational sense, i.e.,*

$$\iint_{\mathbb{R}_+^{n+1}} A(X) \nabla u(X) \overline{\nabla \Phi(X)} dX = \int_{\mathbb{R}^n} g(x) \overline{\varphi(x)} dx,$$

for $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$ and $\varphi := \Phi|_{t=0}$;

(iv) *for the limiting function g from (iii), one has*

$$-e_{n+1} A \nabla u(\cdot, t) \longrightarrow g$$

as $t \rightarrow 0$, in the weak sense in L^p . Here $e_{n+1} = (0, \dots, 0, 1)$. Finally,

$$(2.3) \quad \|g\|_{L^p(\mathbb{R}^n)} \leq C \|\tilde{N}(\nabla u)\|_{L^p(\mathbb{R}^n)}.$$

An analogous statement holds in \mathbb{R}_-^{n+1} .

The Lemma can be found in [HMiMo] (as stated above), [HMM] (for somewhat more general operators), [AAAHK] (for $p = 2$), [KS] (for real coefficients) and the proof in all cases closely follows an analogous argument in [KP].

Remark. We note that the convergence results above entail, in particular, the following. The solvability of (R_p) as defined in the introduction, that is, for $C_0^\infty(\mathbb{R}^n)$ data, implies existence of a solution to (R_p) for any $f \in \dot{L}_1^p(\mathbb{R}^n)$. An analogous statement is valid for the Dirichlet problem. We refer the reader, e.g., to [KS], Theorems 4.6, 5.6, where these results were established for real symmetric matrices, and the proofs apply to our case without changes.

The next order of business is to define a weak solution to the boundary problem $Lu = 0$ in \mathbb{R}_+^{n+1} , $u|_{\mathbb{R}^n} = f \in C_0^\infty(\mathbb{R}^n)$. In principle, taking a harmonic extension of f to \mathbb{R}_+^{n+1} (denote it by w) and then using the Lax-Milgram lemma to resolve $Lu = -Lw$ with zero trace on the boundary, we get a solution in $\dot{W}^{1,2}(\mathbb{R}_+^{n+1})$, the factor space of functions modulo constants with the seminorm given by the norm of the gradient in $L^2(\mathbb{R}_+^{n+1})$. It is somewhat more convenient though to use a non-homogeneous space. One option (and here we follow an approach in [KR]) is to

work in the following framework. Let $\tilde{W}^{1,2}(\mathbb{R}_+^{n+1})$ denote the space of functions F for which

$$\|F\|_{\tilde{W}^{1,2}(\mathbb{R}_+^{n+1})} := \left(\iint_{\mathbb{R}_+^{n+1}} |F(X)|^2 \frac{dX}{1+|X|^2} + \iint_{\mathbb{R}_+^{n+1}} |\nabla F(X)|^2 dX \right)^{1/2} < \infty.$$

One can define the trace operator, for instance, as $\text{Tr} : \tilde{W}^{1,2}(\mathbb{R}_+^{n+1}) \rightarrow \tilde{L}^2(\mathbb{R}^n)$, a continuous extension of the restriction to the boundary operator, with $\tilde{L}^2(\mathbb{R}^n)$ denoting the space of functions f on \mathbb{R}^n with

$$\|f\|_{\tilde{L}^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{1+|x|^2} \right)^{1/2} < \infty,$$

(follow, e.g., the argument in [E], p. 272). Notation $\tilde{W}_0^{1,2}(\mathbb{R}_+^{n+1})$ stands for the space of functions in $\tilde{W}^{1,2}(\mathbb{R}_+^{n+1})$ with trace zero.

Lemma 2.4. *For any $2 < r < \frac{2(n+1)}{n-1}$ and $1 < q < \frac{rn}{n+1}$, $n \geq 1$, and for any $f \in L_1^2(\mathbb{R}^n) \cap L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ there exists a unique $u \in \tilde{W}^{1,2}(\mathbb{R}_+^{n+1})$ such that $\text{Tr } u = f$ and $Lu = 0$, in the usual weak sense. Moreover,*

$$(2.5) \quad \|u\|_{\tilde{W}^{1,2}(\mathbb{R}_+^{n+1})} \lesssim \|f\|_{L_1^2(\mathbb{R}^n)} + \|f\|_{L^r(\mathbb{R}^n)} + \|f\|_{L^q(\mathbb{R}^n)}.$$

Proof. The proof is a modification of an analogous arguments [KR]. Here we only mention the main idea. The remaining details are quite easy to fill in, and if needed, the reader may consult [KR].

The basic idea, already mentioned above, is to realize u as $v + w$, where w is the solution to the Laplace's equation with data f , that is, the Poisson extension of f , and v is the solution to $Lv = G$ (where $G = -Lw$) with zero boundary data, given by the Lax-Milgram Lemma. A direct computation shows that $w \in \tilde{W}^{1,2}(\mathbb{R}_+^{n+1})$ satisfies (2.5). The Lax-Milgram lemma assures existence of the unique solution $v \in \tilde{W}_0^{1,2}(\mathbb{R}_+^{n+1})$ to the problem

$$\iint_{\mathbb{R}_+^{n+1}} A \nabla v \cdot \overline{\nabla \Phi} dx dt = \iint_{\mathbb{R}_+^{n+1}} A \nabla w \cdot \overline{\nabla \Phi} dx dt, \quad \text{for all } \Phi \in \tilde{W}_0^{1,2}(\mathbb{R}_+^{n+1}).$$

This requires boundedness and coercivity of the bilinear form with respect to the norm in $\tilde{W}^{1,2}(\mathbb{R}_+^{n+1})$. Boundedness is obvious from the definition, and the coercivity follows from the Poincaré inequality. We note for the future reference that the Lax-Milgram lemma, in particular, assures that

$$(2.6) \quad \|\nabla v\|_{L^2(\mathbb{R}_+^{n+1})} \leq C \|\nabla w\|_{L^2(\mathbb{R}_+^{n+1})}.$$

Finally, the desired estimates (2.5) follow from the combination of aforementioned bounds on w and (2.6). \square

Let us now provide somewhat more precise asymptotic estimates on the weak solution u constructed above in the case when f is a nice compactly supported function.

Lemma 2.7. *For any $f \in C_0^\infty(\mathbb{R}^n)$ the weak solution $u \in \tilde{W}^{1,2}(\mathbb{R}_+^{n+1})$ to the problem $Lu = 0$, $u|_{\mathbb{R}^n} = f$, warranted by Lemma 2.4, satisfies*

$$(2.8) \quad |u(x, t)| \leq C_f t^{1-\frac{n+1}{2}}, \quad (x, t) \in \mathbb{R}_+^{n+1},$$

provided that t is significantly bigger than the size of the support of f . The constant C depends on f and the operator L .

The statement and the proof of this Lemma is the only place in the paper where constants denoted by C are allowed to depend on data f . The results, however, will be only used qualitatively to ensure the convergence of some later arising integrals.

Proof. Recall the construction of the solution from the proof of Lemma 2.4. The classical harmonic Poisson extension of f , denoted by w , clearly satisfies (2.8). In fact, it decays faster, as $w(X) = \underline{O}(|X|^{-n})$, at infinity. It remains to estimate v . To this end, we decompose v as a sum $\sum_{i=0}^{\infty} v_i$, where $v_i \in \tilde{W}_0^{1,2}(\mathbb{R}_+^{n+1})$ is the Lax-Milgram solution to

$$\iint_{\mathbb{R}_+^{n+1}} A \nabla v_i \overline{\nabla \Phi} dx dt = \iint_{\mathbb{R}_+^{n+1}} A \nabla w_i \overline{\nabla \Phi} dx dt, \quad \text{for all } \Phi \in \tilde{W}_0^{1,2}(\mathbb{R}_+^{n+1}).$$

Here $w_i = \eta_i w$, with η_i , $i = 0, 1, \dots$, being the elements of the usual partition of identity associated to dyadic annuli of radius 2^i centered at the origin (and η_0 associated to the unit ball). Then, in particular, for all i sufficiently large depending on the size of the support of f we have

$$(2.9) \quad \|\nabla v_i\|_{L^2(\mathbb{R}_+^{n+1})} \leq C \|\nabla w_i\|_{L^2(\mathbb{R}_+^{n+1})} \leq C 2^{-i(n+1)/2},$$

using asymptotic estimates for the harmonic functions.

Next, observing that $v_i = 0$ on the boundary we employ Poincaré inequality to write

$$\left(\iint_{B_k(O)} |v_i|^2 dX \right)^{1/2} \leq C 2^k \left(\iint_{B_k(O)} |\nabla v_i|^2 dX \right)^{1/2} \leq C 2^{k(1-\frac{n+1}{2})} 2^{-i(n+1)/2}.$$

Finally, by De Giorgi-Nash-Moser estimates,

$$\begin{aligned} |u(x, t)| &\lesssim \left(\iint_{B_{t/2}(x, t)} |u|^2 dX \right)^{1/2} \leq \left(\iint_{B_{t/2}(x, t)} |w|^2 dX \right)^{1/2} + \sum_{i=1}^{\infty} \left(\iint_{B_{t/2}(x, t)} |v_i|^2 dX \right)^{1/2} \\ &\lesssim t^{-n} + t^{1-\frac{n+1}{2}}, \end{aligned}$$

with the implicit constant depending on f and assuming that t is large enough compared to the size of the support of f . \square

As discussed above, any $u \in \tilde{W}^{1,2}(\mathbb{R}_+^{n+1})$ has a trace in $\tilde{L}^2(\mathbb{R}^n)$. Moreover, for any u which is a solution to $Lu = 0$ in the sense of Lemma 2.4, one can define the conormal derivative $\partial_\nu u = \partial_{\nu_A} u$, in the sense of distributions, via

$$(2.10) \quad \int_{\mathbb{R}^n} \partial_\nu u(x) \overline{\varphi(x)} dx := \iint_{\mathbb{R}_+^{n+1}} A(X) \nabla u(X) \overline{\nabla \Phi(X)} dX,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\Phi \in C_0^\infty(\mathbb{R}_+^{n+1})$ such that $\varphi := \Phi|_{t=0}$. The details are as follows.

Lemma 2.11. *Suppose that $u \in \dot{W}^{1,2}(\mathbb{R}_+^{n+1})$, and that $Lu = 0$ in \mathbb{R}_+^{n+1} . Then $\partial_{\nu_A} u$ exists in $\dot{L}_{-1/2}^2(\mathbb{R}^n)$; i.e., there is a $g_u \in \dot{L}_{-1/2}^2(\mathbb{R}^n)$ such that for all $H \in \dot{W}^{1,2}(\mathbb{R}_+^{n+1})$, with trace $\text{tr}(H) = h \in \dot{L}_{1/2}^2(\mathbb{R}^n)$, we have*

$$(2.12) \quad \iint_{\mathbb{R}_+^{n+1}} A \nabla u \cdot \overline{\nabla H} = \langle g_u, \bar{h} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes here the duality pairing of $\dot{L}_{-1/2}^2(\mathbb{R}^n)$ and $\dot{L}_{1/2}^2(\mathbb{R}^n)$. The analogous statements hold for the adjoint L^* , and in the lower half-space.

Proof. We define a bounded linear functional Λ_u on $\dot{L}_{1/2}^2(\mathbb{R}^n)$ as follows. For $h \in \dot{L}_{1/2}^2(\mathbb{R}^n)$, set

$$\Lambda_u(h) := \iint_{\mathbb{R}_+^{n+1}} A \nabla u \cdot \overline{\nabla H},$$

where H is any $\dot{W}^{1,2}(\mathbb{R}_+^{n+1})$ extension of h (of course, such extensions exist by standard extension/trace theory). Note that Λ_u is well-defined: indeed, if H_1 and H_2 are two different $\dot{W}^{1,2}(\mathbb{R}_+^{n+1})$ extensions of the same h , then $H_1 - H_2 \in \dot{W}_0^{1,2}(\mathbb{R}_+^{n+1})$, whence it follows that

$$\iint_{\mathbb{R}_+^{n+1}} A \nabla u \cdot \overline{\nabla(H_1 - H_2)} = 0,$$

since $Lu = 0$ in the weak sense. Moreover, it is obvious that Λ_u is linear. To see that the functional is bounded, we simply choose an extension H (e.g., the harmonic extension), for which

$$\|H\|_{\dot{W}^{1,2}(\mathbb{R}_+^{n+1})} \leq C_0 \|h\|_{\dot{L}_{1/2}^2(\mathbb{R}^n)},$$

for some purely dimensional constant C_0 . We then have

$$|\Lambda_u(h)| \leq \|A\|_\infty \|u\|_{\dot{W}^{1,2}(\mathbb{R}_+^{n+1})} \|H\|_{\dot{W}^{1,2}(\mathbb{R}_+^{n+1})} \leq C_0 \|A\|_\infty \|u\|_{\dot{W}^{1,2}(\mathbb{R}_+^{n+1})} \|h\|_{\dot{L}_{1/2}^2(\mathbb{R}^n)},$$

i.e., $\|\Lambda_u\| \lesssim \|u\|_{\dot{W}^{1,2}(\mathbb{R}_+^{n+1})}$. The conclusion of Lemma 2.11 now follows by the Riesz Representation Theorem. \square

Lemma 2.11 allows us to justify the definition of the conormal derivative by (2.10) for all functions $u \in \dot{W}^{1,2}(\mathbb{R}_+^{n+1})$, by first identifying such a function with the corresponding equivalence class in $\dot{W}^{1,2}(\mathbb{R}_+^{n+1})$, then applying Lemma 2.11 and then deducing that for any particular representative of this equivalence class, in particular, for u itself, we have (2.10). Such a definition gives identical result for u in the same equivalence class, but, naturally, the conormal derivative would not distinguish functions that differ by a constant.

Note that if, in addition, $\tilde{N}(\nabla u) \in L^p(\mathbb{R}^n)$, for some $1 < p < \infty$, then $\partial_{\nu_A} u \in L^p(\mathbb{R}^n)$ by Lemma 2.2.

3. BOUNDARY ESTIMATE: A VERSION OF THE RELICH-TYPE INEQUALITY

Theorem 3.1. *Let L be an elliptic operator with t -independent coefficients such that the solutions to $Lu = 0$ and $L^*u = 0$ in \mathbb{R}_\pm^{n+1} satisfy the De Giorgi-Nash-Moser estimates. Let u be a solution to the Dirichlet problem $Lu = 0$ in \mathbb{R}_+^{n+1} , $u|_{\partial\mathbb{R}_+^{n+1}} = f$, for some $f \in C_0^\infty(\mathbb{R}^n)$, in the sense of Lemma 2.4. Suppose that for*

some $1 < p' < \infty$ the Dirichlet problem $(D_{p'})$ for the operator L^* is solvable with the square function bounds, that is, the solution u satisfies both (1.4) and the square function estimate (1.12). Then the variational derivative of u defined by (2.10) can be identified with an L^p function, and

$$(3.2) \quad \|\partial_{\nu_A} u\|_{L^p(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. The constant C depends on the standard constants and on the solvability constants of L^* involved in (1.4) and (2.10).

Proof. We aim to show that for u , a solution to $Lu = 0$ in \mathbb{R}_+^{n+1} , the normal derivative on the boundary is controlled by the tangential derivatives for some $1 < p < \infty$. To this end, take $1 < p' < \infty$ such that $(D_{p'})$ for the operator L^* is solvable with the square function bounds, consider any $g \in C_0^\infty$ with $\|g\|_{L^{p'}} \leq 1$, and denote by w a solution to $(D_{p'})$ for the operator L^* with boundary data g . Then

$$(3.3) \quad \int_{\mathbb{R}^n} \partial_{\nu_A} u \bar{g} dx = \iint_{\mathbb{R}_+^{n+1}} \nabla F(X) \overline{A^*(X) \nabla w(X)} dX = \int_{\mathbb{R}^n} f \overline{\partial_{\nu_{A^*}} w} dx,$$

for any $F \in C_0^\infty(\mathbb{R}^{n+1})$ such that $F|_{\mathbb{R}^n} = f$. This follows simply from the definition of the weak conormal derivative in (2.10). Indeed, by definition

$$\int_{\mathbb{R}^n} f \overline{\partial_{\nu_{A^*}} w} dx = \iint_{\mathbb{R}_+^{n+1}} \nabla F(X) \overline{A^*(X) \nabla w(X)} dX = \iint_{\mathbb{R}_+^{n+1}} \nabla u(X) \overline{A^*(X) \nabla w(X)} dX,$$

where for the second equality we use the fact that $u - F \in \tilde{W}_0^{1,2}(\mathbb{R}_+^{n+1})$ and w is a solution in the sense of (1.2) (evidently, C_0^∞ functions are dense in $\tilde{W}_0^{1,2}(\mathbb{R}_+^{n+1})$). A similar argument applies to show that

$$\int_{\mathbb{R}^n} \partial_{\nu_A} u g dx = \iint_{\mathbb{R}_+^{n+1}} A(X) \nabla u(X) \overline{\nabla w(X)} dX.$$

We shall take an extension of f in the form $F(x, t) := f(x) \eta_{r,R}(t)$, $R \gg r$, where $\eta_{r,R}(t) \in C_0^\infty((-R+r, R+r))$, $\eta_{r,R}(t) = 1$ for $t \in (-R, R)$, and $|\eta'| \leq 1/r$. Here r is chosen so that Δ_r contains $\text{supp}(f)$. Then the right-hand side of (3.3) is equal to

$$(3.4) \quad \begin{aligned} & \iint_{\Delta_r \times (0, R)} \nabla F(X) \overline{A^*(X) \nabla w(X)} dX + \iint_{\Delta_r \times (R, r+R)} \nabla F(X) \overline{A^*(X) \nabla w(X)} dX \\ &= \sum_{j=1}^n \iint_{\Delta_r \times (0, R)} \partial_j f(x) e_j \overline{A^*(X) \nabla w(X)} dX + \iint_{\Delta_r \times (R, r+R)} \nabla F(X) \overline{A^*(X) \nabla w(X)} dX \\ &= \sum_{j=1}^n \int_{\Delta_r} \partial_j f(x) \left(\int_0^R e_j \overline{A^*(x) \nabla w(x, t)} dt \right) dx + \iint_{\Delta_r \times (R, r+R)} \nabla F(X) \overline{A^*(X) \nabla w(X)} dX, \end{aligned}$$

where $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ is the j -th basis vector of \mathbb{R}^{n+1} , and we used the fact that by construction $F = f$ in $\Delta_r \times (0, R)$ and hence, in this range, it is independent of t .

We remark that, intuitively, one can think of the functions in parentheses above as an analogue of harmonic conjugates. That is, given a solution w to $Lw = 0$ in \mathbb{R}_+^{n+1} , a system of L -harmonic conjugates could be defined as follows:

$$(3.5) \quad \tilde{w}_j(x, t) := - \int_t^\infty e_j A(x) \nabla w(x, s) ds, \quad (x, t) \in \mathbb{R}_+^{n+1}, \quad j = 1, \dots, n+1,$$

(see, e.g., [FS] and [KR] for analogous constructions in the case of harmonic functions and variable-coefficient operators in dimension 2, respectively). Thus, (3.3)–(3.4) and forthcoming calculations are actually manipulations with harmonic conjugates in disguise. However, because of the weak nature of the available definition of solution and conormal derivative, we have to carefully keep track of the error terms.

Going further, departing from the right-hand side of (3.4), we can write

$$(3.6) \quad \begin{aligned} & \int_{\mathbb{R}^n} \partial_{\nu_A} u \bar{g} dx \\ &= \sum_{i,j=1}^n \int_{\Delta_r} \partial_j f(x) \left(\int_0^R \overline{A_{ji}^*(x) \partial_i w(x, t)} dt \right) dx \\ & - \sum_{j=1}^n \int_{\Delta_r} \partial_j f(x) \overline{A_{j,n+1}^*(x) g(x)} dx + \sum_{j=1}^n \int_{\Delta_r} \partial_j f(x) \overline{A_{j,n+1}^*(x) w(x, R)} dx \\ & + \iint_{\Delta_r \times (R, r+R)} \nabla F(X) \overline{A^*(X) \nabla w(X)} dX =: I_R + II + E_{1,R} + E_{2,R}. \end{aligned}$$

First of all, we claim that the terms $E_{1,R}$ and $E_{2,R}$ both vanish as $R \rightarrow \infty$. Indeed,

$$E_{2,R} \leq C_{f,r} \left(\iint_{2\Delta_r \times (R-r, 2r+R)} |w|^2 dX \right)^{1/2} \leq C_{f,r} R^{1-\frac{n+1}{2}},$$

using the Cauchy-Schwarz and Caccioppoli inequalities for the first bound and Lemma 2.7 for the second one. Analogously, using Lemma 2.7, we see that $E_{1,R} \leq C_{f,r} R^{1-\frac{n+1}{2}}$ as well.

It remains to analyze I_R and II . The integral in II directly gives the desired bound by $\|\nabla_{\parallel} f\|_{L^p}$. The estimate on I_R is trickier. Using, as before, the decay of w assured by Lemma 2.7, we see that it is enough to bound

$$(3.7) \quad I := \int_{\mathbb{R}^n} \nabla_{\parallel} f(x) \cdot \overline{A_{\parallel}^*(x) \nabla_{\parallel} v(x)} dx,$$

where formally

$$(3.8) \quad v(x, t) := \int_t^\infty w(x, s) ds, \quad v(x) := v(x, 0),$$

as the error vanishes as $R \rightarrow \infty$. Here and in the sequel, A_{\parallel} denotes an $n \times n$ block of the matrix A , that is, $\{A_{jk}\}_{j,k=1}^n$, and $L_{\parallel} = -\operatorname{div}_{\parallel} A_{\parallel} \nabla_{\parallel}$, interpreted, as usually, in the weak sense.

Let us discuss the definition of v . First of all, $v(x, t)$ itself is well-defined for any $t > 0$ as an absolutely convergent integral (using Lemma 2.7), and $\nabla v(\cdot, t)$ belongs to $L^2_{loc}(\mathbb{R}^n)$ for any $t > 0$. Further, $\nabla_{\parallel} v$ on the boundary is well-defined as an $L^2_{loc}(\mathbb{R}^n)$ function (using the fact that $w \in \tilde{W}^{1,2}(\mathbb{R}^{n+1}_+)$ and Lemma 2.7), and $\nabla_{\parallel} v(\cdot, t)$ converges to $\nabla_{\parallel} v(\cdot)$ in the sense of distributions and in $L^2_{loc}(\mathbb{R}^n)$. This clarifies the sense of (3.7)–(3.8). We note for the future reference that $\partial_t v = w$ belongs to $\tilde{W}^{1,2}(\mathbb{R}^{n+1}_+)$ and satisfies Lemma 2.7, (2.1), and (1.8).

We claim that

$$(3.9) \quad \left| \int_{\mathbb{R}^n} \nabla_{\parallel} f(x) \cdot \overline{A_{\parallel}^*(x) \nabla_{\parallel} v(x)} dx \right| \lesssim \|\nabla_{\parallel} f\|_{L^p} \left(\|\mathcal{A}(t \nabla \partial_t v)\|_{L^{p'}} + \|N_*(\partial_t v)\|_{L^{p'}} \right),$$

where \mathcal{A} , as before, stands for the square function (1.13). Estimate (3.9) is one of the core components of our approach to the regularity problem and we will concentrate on its proof in the next section. It is interesting to point out that the particular quadratic form appearing here is important. Indeed, the case of interest is $p < 2$. The function v itself is a solution to the Dirichlet problem for L^* , and it is not in general true that the derivatives of the solution belong to some $L^{p'}$, $p' > 2$.

For now, let us finish the proof of the Theorem assuming (3.9).

Recall that $\partial_t v = w$ and by definition w is the solution to $(D_{p'})$ for L^* satisfying both the non-tangential maximal function and the square function estimates. All in all, then (3.6)–(3.9) guarantee that for any $g \in L^{p'}$

$$(3.10) \quad \int_{\mathbb{R}^n} \partial_{\nu_A} u \bar{g} dx \lesssim \|\nabla_{\parallel} f\|_{L^p} \|g\|_{L^{p'}}.$$

Hence,

$$(3.11) \quad \|\partial_{\nu_A} u\|_{L^p(\mathbb{R}^n)} \lesssim \|\nabla_{\parallel} f\|_{L^p(\mathbb{R}^n)},$$

with $f = u|_{\mathbb{R}^n}$. This finished the proof of Theorem 3.1, modulo (3.9). \square

4. PROOF OF THE MAIN ESTIMATE (3.9)

In this section, we establish the “main estimate” (3.9) (re-stated as Theorem 4.10 below) thereby completing the proof of Theorem 3.1. We shall adapt the proof of [AAAHK, estimate (5.3)], which is essentially the case $p = 2$ of (3.9), and which exploits the solution of the Kato problem. In our case, we require L^p versions of the Kato estimate (cf. (4.2) below).

Let us start by recalling a few results regarding the square roots of elliptic operators and the corresponding square functions that will be used throughout the proof. Retain the definitions of N_* , \tilde{N} and \mathcal{A} from Sections 1 and 3, and let

$$\mathcal{A}_q^\alpha(F)(x) = \left(\iint_{|x-z| < \alpha t} |F(z, t)|^q \frac{dz dt}{t^{n+1}} \right)^{1/q},$$

$$C_q(F)(x) = \sup_{\Delta \ni x} \left(\frac{1}{|\Delta|} \iint_{\widehat{\Delta}} |F(y, s)|^q \frac{dy ds}{s} \right)^{1/q},$$

where, as before, $\widehat{\Delta} = \{(x, t) : \text{dist}(x, \Delta^c) \geq t\}$ and $\Delta(x, t) = \{y \in \mathbb{R}^n : |x - y| < t\}$. The aperture index α will usually be omitted unless it plays an explicit role in the proof. Also, as per (1.13), $\mathcal{A} = \mathcal{A}_2$, and $C = C_2$.

For a Lebesgue measurable set E , we let $\mathbf{M}(E)$ denote the collection of measurable functions on E . For $0 < p, q < \infty$ we define the following tent spaces:

$$\begin{aligned} T_q^p(\mathbb{R}_+^{n+1}) &= \{F \in \mathbf{M}(\mathbb{R}_+^{n+1}) : \mathcal{A}_q(F) \in L^p(\mathbb{R}^n)\}, \\ T_\infty^p(\mathbb{R}_+^{n+1}) &= \{F \in \mathbf{M}(\mathbb{R}_+^{n+1}) : N_*(F) \in L^p(\mathbb{R}^n)\}, \\ \widetilde{T}_\infty^p(\mathbb{R}_+^{n+1}) &= \{F \in \mathbf{M}(\mathbb{R}_+^{n+1}) : \widetilde{N}(F) \in L^p(\mathbb{R}^n)\}, \\ T_q^\infty(\mathbb{R}_+^{n+1}) &= \{F \in \mathbf{M}(\mathbb{R}_+^{n+1}) : C_q(F) \in L^\infty(\mathbb{R}^n)\}, \\ \mathbf{T}_q^\infty(\mathbb{R}_+^{n+1}) &= \{F \in \mathbf{M}(\mathbb{R}_+^{n+1}) : C_q(\mathbf{s}(F)) \in L^\infty(\mathbb{R}^n)\}, \end{aligned}$$

where in the last definition, for $F \in \mathbf{M}(\mathbb{R}_+^{n+1})$, we set

$$\mathbf{s}(F)(x, t) := \sup_{(y, s) \in W(x, t)} |F(y, s)|$$

(the notation “sup” is interpreted as the essential supremum) and $W(x, t) = \Delta(x, t) \times (t/2, 3t/2)$. The spaces $T_q^p(\mathbb{R}_+^{n+1})$, $0 < p, q \leq \infty$ were first introduced by Coifman, Meyer and Stein in [CMS]. The spaces $\widetilde{T}_\infty^p(\mathbb{R}_+^{n+1})$ and $\mathbf{T}_q^\infty(\mathbb{R}_+^{n+1})$ started appearing in the literature more recently, naturally arising for elliptic PDEs with non-smooth coefficients.

As usual, we say that a family of operators $\{T_t\}_{t>0}$ satisfies $L^p - L^q$ off-diagonal estimates, $1 \leq p \leq q \leq \infty$, if for arbitrary closed sets $E, F \subset \mathbb{R}^n$

$$(4.1) \quad \|T_t f\|_{L^q(F)} \leq C t^{\left(\frac{n}{q} - \frac{n}{p}\right)} e^{-\frac{\text{dist}(E, F)^2}{ct}} \|f\|_{L^p(E)},$$

for every $t > 0$ and every $f \in L^p(\mathbb{R}^n)$ supported in E . We remark that whenever L and L^* both satisfy the De Giorgi-Nash-Moser property, the heat semigroup $\mathcal{P}_t := e^{-t^2 L_\parallel}$, $t > 0$, satisfies pointwise Gaussian upper bounds, and hence, $L^p - L^q$ off-diagonal estimates for all $1 \leq p, q \leq \infty$. Indeed, if the solutions of L have the De Giorgi-Nash-Moser bounds, then so do the solutions to L_\parallel . To see that, let $\Delta = \Delta(x, r)$ be an n -dimensional ball, and let $B = B((x, r), r/2)$ be the corresponding $(n+1)$ -dimensional ball. Let $u = u(x)$ solve $L_\parallel u = 0$ in 2Δ . Set $U(x, t) := u(x)$, so that U is t -independent. Since U and also the coefficients of L are t -independent, we have that $LU(x, t) = L_\parallel u(x) = 0$ for all $(x, t) \in 2B$. Since U satisfies the De Giorgi-Nash-Moser property in B , we conclude that u satisfies the De Giorgi-Nash-Moser property in Δ . Furthermore, if the solutions to both L_\parallel and L_\parallel^* have the De Giorgi-Nash-Moser bounds, then the heat kernel, that is, the kernel of the semigroup $\mathcal{P}_t := e^{-t^2 L_\parallel}$, $t > 0$, satisfies pointwise Gaussian upper bounds, and enjoys Nash type local Hölder continuity, by [AT] (Theorem 10, p. 34, *loc. cit.*), as desired.

The latter, in turn, imply that the square root estimate,

$$(4.2) \quad \left\| \sqrt{L_{\parallel}} f \right\|_{L^p} \leq C \|\nabla f\|_{L^p},$$

holds for all $1 < p < \infty$, $f \in C_0^\infty$, (the case $p = 2$ corresponds to the Kato problem solved in [AHLMcT], and the generalization to other values of p (given the L^2 case), in the presence of Gaussian bounds, can be found in [AT]). The Gaussian bounds also imply that

$$(4.3) \quad \|\mathcal{A}(\theta_t f)\|_{L^p} \leq C \|f\|_{L^p},$$

$$\text{with } \theta_t = t \sqrt{L_{\parallel}} \mathcal{P}_t, \theta_t = t^2 \nabla_{\parallel} \sqrt{L_{\parallel}} \mathcal{P}_t, \theta_t = t^3 L_{\parallel}^{3/2} \mathcal{P}_t, \theta_t = t^4 \nabla_{\parallel} L_{\parallel}^{3/2} \mathcal{P}_t,$$

holds for all $1 < p < \infty$, $f \in L^p$. In the required generality these square function estimates do not seem to be explicitly stated anywhere, but they are all essentially well known. Indeed, one may verify the case $p = 2$ by a standard “quasi-orthogonality” argument; for $p > 2$, one may follow the well known argument of [FS] to get a Carleson measure estimate when $f \in L^\infty$, and then use tent space interpolation to obtain all $p \in [2, \infty)$; for $p < 2$, one may first prove the Hardy space bound $\|\mathcal{A}(\theta_t f)\|_{L^1} \leq C \|f\|_{H^1}$, by a standard argument using the atomic decomposition of H^1 , and the local Hölder continuity of the heat kernel, and then interpolate to get the full range of p . We omit the details.

Going further, we record the following result essentially following from the Poincaré inequality. It was proved for $p = 2$ in [AAAHK] (Lemma 3.5, *loc. cit.*).

Lemma 4.4. *Assume that a family of operators $\{R_t\}_{t>0}$ satisfies L^2 – L^2 off-diagonal estimates and that $R_t 1 = 0$ (in the sense of $L^2_{\text{loc}}(\mathbb{R}^n)$). Then*

$$(4.5) \quad \|\mathcal{A}(R_t F)\|_{L^p} \leq C \|\mathcal{A}(t \nabla_{\parallel} F)\|_{L^p},$$

for every F with $t \nabla_{\parallel} F \in T_2^p$, i.e., such that the right-hand side of (4.5) is finite, and for every $1 < p < \infty$.

In fact, a weaker off-diagonal decay rate of R_t than the exponential estimates above would suffice, but for all relevant choices of R_t ’s in the present paper the exponential decay will be valid.

Proof. Let us denote $\Delta_{x,t} = \{y \in \mathbb{R}^n : |x - y| < t\}$, $S_j(\Delta_{x,t}) = \Delta_{x,2^{j+1}t} \setminus \Delta_{x,2^j t}$, $j \in \mathbb{N}$, and $F_{x,t} := \int_{\Delta_{x,t}} F(y, t) dy$. Then

$$\begin{aligned} (4.6) \quad \|\mathcal{A}^1(R_t F)\|_{L^p} &= \left(\int_{\mathbb{R}^n} \left(\iint_{|x-y|<t} |R_t F(y, t)|^2 \frac{dy dt}{t^{n+1}} \right)^{p/2} dx \right)^{1/p} \\ &\leq \left(\int_{\mathbb{R}^n} \left(\iint_{|x-y|<t} |R_t([F(\cdot, t) - F_{x,2t}]\chi_{\Delta_{x,2t}})(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{p/2} dx \right)^{1/p} \\ &\quad + \sum_{j=1}^{\infty} \left(\int_{\mathbb{R}^n} \left(\iint_{|x-y|<t} |R_t([F(\cdot, t) - F_{x,2t}]\chi_{S_j(\Delta_{x,t})})(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{p/2} dx \right)^{1/p} \\ &= I + II. \end{aligned}$$

Using the uniform in t boundedness of R_t in $L^2(\mathbb{R}^n)$ (following from the $L^2 - L^2$ off-diagonal estimates) and then the Poincaré inequality, we deduce that

$$(4.7) \quad I \lesssim \left(\int_{\mathbb{R}^n} \left(\iint_{|x-y| < 2t} |F(y, t) - F_{x, 2t}|^2 \frac{dy dt}{t^{n+1}} \right)^{p/2} dx \right)^{1/p} \\ \lesssim \left(\int_{\mathbb{R}^n} \left(\iint_{|x-y| < 2t} |t \nabla_{\parallel} F(y, t)|^2 \frac{dy dt}{t^{n+1}} \right)^{p/2} dx \right)^{1/p} = \|\mathcal{A}^2(t \nabla_{\parallel} F)\|_{L^p}.$$

On the other hand, by $L^2 - L^2$ off-diagonal estimates,

$$(4.8) \quad II \lesssim \sum_{j=1}^{\infty} \left(\int_{\mathbb{R}^n} \left(\int_0^{\infty} e^{-(2^j t)^2 / (c t^2)} \int_{S_j(\Delta_{x, t})} |F(y, t) - F_{x, 2t}|^2 dy \frac{dt}{t^{n+1}} \right)^{p/2} dx \right)^{1/p} \\ \lesssim \sum_{j=1}^{\infty} \sum_{k=1}^j \left(\int_{\mathbb{R}^n} \left(\int_0^{\infty} e^{-C 4^j} \left(\frac{2^j}{2^k} \right)^n \int_{\Delta_{x, 2^{k+1}t}} |F(y, t) - F_{x, 2^{k+1}t}|^2 dy \frac{dt}{t^{n+1}} \right)^{p/2} dx \right)^{1/p},$$

where we used the representation

$$F(y, t) - F_{x, 2t} = F(y, t) - F_{x, 2^{j+1}t} + F_{x, 2^{j+1}t} - F_{x, 2^j t} + \dots + F_{x, 2^2 t} - F_{x, 2t}$$

for the second inequality. Using now the Poincaré inequality, we have

$$(4.9) \quad II \lesssim \sum_{j=1}^{\infty} \sum_{k=1}^j \left(\int_{\mathbb{R}^n} \left(\int_0^{\infty} e^{-C 4^j} 2^{jn} 2^{-kn} 2^{2k} \int_{\Delta_{x, 2^{k+1}t}} |t \nabla_{\parallel} F(y, t)|^2 dy \frac{dt}{t^{n+1}} \right)^{p/2} dx \right)^{1/p} \\ \lesssim \sum_{k=1}^{\infty} 2^{-Mk} \left(\int_{\mathbb{R}^n} \left(\int_0^{\infty} \int_{\Delta_{x, 2^{k+1}t}} |t \nabla_{\parallel} F(y, t)|^2 dy \frac{dt}{t^{n+1}} \right)^{p/2} dx \right)^{1/p},$$

where $M > n/2$ can be arbitrarily large constant. However, (4.9) simply says that

$$II \lesssim \sum_{k=1}^{\infty} 2^{-Mk} \left\| \mathcal{A}^{2^{k+1}}(t \nabla_{\parallel} F) \right\|_{L^p},$$

which in turn implies that

$$I + II \lesssim \sum_{k=0}^{\infty} 2^{-Mk + C(n, p)k} \|\mathcal{A}^1(t \nabla_{\parallel} F)\|_{L^p} \lesssim \|\mathcal{A}^1(t \nabla_{\parallel} F)\|_{L^p},$$

since M can always be taken large enough. We remark that the sharp constant $C(n, p)$ appearing in the change-of-aperture square function estimates was obtained in [A2], although for the purposes of the present argument we only need to know

that the dependence on the aperture is polynomial, and this was already established in [CMS]. \square

At this point we are ready to turn to the proof of estimate (3.9).

Theorem 4.10. *Assume that L is an elliptic operator with t -independent coefficients, and that L and L^* satisfy the De Giorgi-Nash-Moser bounds. Let w be the weak solution to the Dirichlet problem for L^* with some C_0^∞ data, guaranteed by Lemma 2.4, and v be its antiderivative, defined by (3.8). Then*

$$(4.11) \quad \left| \int_{\mathbb{R}^n} \nabla_{\parallel} f(x) \cdot \overline{A_{\parallel}^*(x) \nabla_{\parallel} v(x, 0)} dx \right| \leq C \|\nabla_{\parallel} f\|_{L^p} \left(\|\mathcal{A}(t \nabla(\partial_t v))\|_{L^{p'}} + \|N_*(\partial_t v)\|_{L^{p'}} \right)$$

for every $f \in C_0^\infty(\mathbb{R}^n)$ and $1 < p < \infty$. Here $C > 0$ depends on the standard constants only.

Clearly, (4.11) is only of interest when the right-hand side of (4.11) is finite. It is the case, e.g., when the Dirichlet problem is solvable in $L^{p'}$, and the solution satisfies the square function estimates. However, this information is not needed to establish (4.11).

Remark. For future reference, we point out that the argument will establish a more general result. To be specific, for L and f as in the statement of Theorem 4.10 we shall demonstrate that

$$(4.12) \quad \left| \int_{\mathbb{R}^n} \nabla_{\parallel} f(x) \cdot \overline{A_{\parallel}^*(x) \nabla_{\parallel} v(x, 0)} dx \right| \leq C \|\nabla_{\parallel} f\|_{L^p} \left(\|\mathcal{A}(t \nabla(\partial_t v))\|_{L^{p'}} + \|N_*(\partial_t v)\|_{L^{p'}} \right) + \left| \iint_{\mathbb{R}_+^{n+1}} \partial_t^2 \mathcal{P}_t f(x) \overline{L^* v(x, t)} t dt dx \right|,$$

for every $v : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}$ with reasonable decay properties sufficient to justify convergence of involved integrals. Here $\mathcal{P}_t := e^{-t^2 L_{\parallel}}$, $t > 0$, is, as before, the heat semigroup associated to the operator L_{\parallel} . Clearly, when v is a solution, as in the statement of Theorem 4.10, the last integral on the right hand side of (4.12) is equal to zero and (4.12) reduces to (4.11).

Proof of Theorem 4.10. To begin,

$$(4.13) \quad \begin{aligned} & \int_{\mathbb{R}^n} \nabla_{\parallel} f(x) \cdot \overline{A_{\parallel}^*(x) \nabla_{\parallel} v(x, 0)} dx \\ &= - \iint_{\mathbb{R}_+^{n+1}} \frac{\partial}{\partial t} \left(\nabla_{\parallel} \mathcal{P}_t f(x) \cdot \overline{A_{\parallel}^*(x) \nabla_{\parallel} v(x, t)} \right) dt dx \\ &= - \iint_{\mathbb{R}_+^{n+1}} \nabla_{\parallel} \partial_t \mathcal{P}_t f(x) \cdot \overline{A_{\parallel}^*(x) \nabla_{\parallel} v(x, t)} dt dx - \iint_{\mathbb{R}_+^{n+1}} \nabla_{\parallel} \mathcal{P}_t f(x) \cdot \overline{A_{\parallel}^*(x) \nabla_{\parallel} \partial_t v(x, t)} dt dx \\ &= \iint_{\mathbb{R}_+^{n+1}} \nabla_{\parallel} \mathcal{P}_t f(x) \cdot \overline{A_{\parallel}^*(x) \nabla_{\parallel} \partial_t^2 v(x, t)} t dt dx \end{aligned}$$

$$\begin{aligned}
& + 2 \iint_{\mathbb{R}_+^{n+1}} \nabla_{\parallel} \partial_t \mathcal{P}_t f(x) \cdot \overline{A_{\parallel}^*(x) \nabla_{\parallel} \partial_t v(x, t)} t dt dx \\
& + \iint_{\mathbb{R}_+^{n+1}} \nabla_{\parallel} \partial_t^2 \mathcal{P}_t f(x) \cdot \overline{A_{\parallel}^*(x) \nabla_{\parallel} v(x, t)} t dt dx =: I + II + III.
\end{aligned}$$

The convergence of integrals and integration by parts above and through the argument is justified by our assumptions on v (in particular, the properties of v and $\partial_t v$ outlined in the paragraph above (3.9)), and off-diagonal estimates for \mathcal{P}_t and its derivatives.

Then

$$\begin{aligned}
(4.14) \quad |I| &= \left| \iint_{\mathbb{R}_+^{n+1}} L_{\parallel} \mathcal{P}_t f(x) \overline{\partial_t^2 v(x, t)} t dt dx \right| \\
&\lesssim \left\| \mathcal{A}(t \sqrt{L_{\parallel}} \mathcal{P}_t(\sqrt{L_{\parallel}} f)) \right\|_{L^p} \left\| \mathcal{A}(t \partial_t^2 v) \right\|_{L^{p'}} \lesssim \|\nabla_{\parallel} f\|_{L^p} \left\| \mathcal{A}(t \partial_t^2 v) \right\|_{L^{p'}},
\end{aligned}$$

using boundedness of the square function based on $t \sqrt{L_{\parallel}} \mathcal{P}_t$ (4.3) and then the Kato estimate (4.2), both available for $1 < p < \infty$. Largely by the same argument,

$$\begin{aligned}
(4.15) \quad |II| &\lesssim \left\| \mathcal{A}(t^2 \nabla_{\parallel} \sqrt{L_{\parallel}} \mathcal{P}_t(\sqrt{L_{\parallel}} f)) \right\|_{L^p} \left\| \mathcal{A}(t \nabla_{\parallel} \partial_t v) \right\|_{L^{p'}} \\
&\lesssim \|\nabla_{\parallel} f\|_{L^p} \left\| \mathcal{A}(t \nabla_{\parallel} \partial_t v) \right\|_{L^{p'}}.
\end{aligned}$$

The estimate on III is more delicate. We write

$$\begin{aligned}
(4.16) \quad III &= \iint_{\mathbb{R}_+^{n+1}} \partial_t^2 \mathcal{P}_t f(x) \overline{L^* v(x, t)} t dt dx \\
&+ \sum_{j=1}^n \iint_{\mathbb{R}_+^{n+1}} \partial_t^2 \mathcal{P}_t f(x) \overline{\partial_j A_{j,n+1}^* \partial_t v(x, t)} t dt dx \\
&+ \sum_{j=1}^{n+1} \iint_{\mathbb{R}_+^{n+1}} \partial_t^2 \mathcal{P}_t f(x) \overline{A_{n+1,j}^* \partial_j \partial_t v(x, t)} t dt dx =: III_1 + III_2 + III_3.
\end{aligned}$$

The term III_1 is left alone for the moment. It is zero when v is a solution and otherwise, it shows up explicitly on the right hand side of (4.12). The term III_3 can be handled just as I and II above and gives the bound

$$(4.17) \quad III_3 \lesssim \|\nabla_{\parallel} f\|_{L^p} \left\| \mathcal{A}(t \nabla \partial_t v) \right\|_{L^{p'}},$$

once again using (4.3) and (4.2).

As for III_2 ,

$$\begin{aligned}
(4.18) \quad III_2 &= C \sum_{j=1}^n \iint_{\mathbb{R}_+^{n+1}} \partial_j \partial_t^2 \mathcal{P}_t f(x) \overline{A_{j,n+1}^* (I - P_t) \partial_t v(x, t)} t dt dx \\
&+ C \sum_{j=1}^n \iint_{\mathbb{R}_+^{n+1}} \partial_t^2 \mathcal{P}_t f(x) \overline{(\partial_j A_{j,n+1}^* P_t \partial_t v(x, t))} t dt dx =: III_{2,1} + III_{2,2},
\end{aligned}$$

where P_t is a nice approximation of identity, e.g., the heat semigroup of the Laplacian. Since $R_t := I - P_t$ kills constants, and, by virtue of standard heat kernel bounds, satisfies off-diagonal decay estimates, one can use the Poincaré inequality to produce the gradient and get an estimate on $III_{2,1}$ akin to that for $I + II$. Indeed, according to Lemma 4.4, square function estimates (4.3) and Kato estimate (4.2),

$$III_{2,1} \lesssim \|\sqrt{L_{\parallel}}f\|_{L^p} \|\mathcal{A}((I - P_t)\partial_t v)\|_{L^{p'}} \lesssim \|\nabla_{\parallel}f\|_{L^p} \|\mathcal{A}(t\nabla\partial_t v)\|_{L^{p'}}.$$

Concerning $III_{2,2}$, we further write

$$\begin{aligned} (4.19) \quad III_{2,2} &= C \sum_{j=1}^n \iint_{\mathbb{R}_+^{n+1}} \left(4t^2 L_{\parallel}^2 e^{-t^2 L_{\parallel}} - 2L_{\parallel} e^{-t^2 L_{\parallel}} \right) f(x) \overline{\left(\partial_j A_{j,n+1}^* P_t \partial_t v(x, t) \right)} t dt dx \\ &= \sum_{j=1}^n \iint_{\mathbb{R}_+^{n+1}} \left(C_1 t^2 L_{\parallel}^2 e^{-\frac{3}{4}t^2 L_{\parallel}} + C_2 L_{\parallel} e^{-\frac{3}{4}t^2 L_{\parallel}} \right) f(x) \overline{\left(\partial_j A_{j,n+1}^* P_t \partial_t v(x, t) \right)} t dt dx \\ &= \sum_{j=1}^n \iint_{\mathbb{R}_+^{n+1}} \left(C_1 t^2 L_{\parallel}^2 e^{-\frac{3}{4}t^2 L_{\parallel}} + C_2 L_{\parallel} e^{-\frac{3}{4}t^2 L_{\parallel}} \right) f(x) \times \\ &\quad \times \overline{\left(\mathcal{P}_{t/2}^* \partial_j A_{j,n+1}^* P_t \partial_t v(x, t) - (\mathcal{P}_{t/2}^* \partial_j A_{j,n+1}^*) (P_t \partial_t v(x, t)) \right)} t dt dx \\ &\quad + \sum_{j=1}^n \iint_{\mathbb{R}_+^{n+1}} \left(C_1 t^2 L_{\parallel}^2 e^{-\frac{3}{4}t^2 L_{\parallel}} + C_2 L_{\parallel} e^{-\frac{3}{4}t^2 L_{\parallel}} \right) f(x) \times \\ &\quad \times \overline{(\mathcal{P}_{t/2}^* \partial_j A_{j,n+1}^*) (P_t \partial_t v(x, t))} t dt dx = III'_{2,2} + III''_{2,2}, \end{aligned}$$

where $(\mathcal{P}_{t/2}^* \partial_j A_{j,n+1}^*) (P_t \partial_t v(x, t))$ is interpreted as a product of two functions, while $\mathcal{P}_{t/2}^* \partial_j A_{j,n+1}^* P_t \partial_t v(x, t)$, as before, is a result of an operator $\mathcal{P}_{t/2}^* \partial_j A_{j,n+1}^* P_t$ acting on the function $\partial_t v$.

Much as in the analysis of $III_{2,1}$, we note that the operator

$$R_t := t \left(\mathcal{P}_{t/2}^* \partial_j A_{j,n+1}^* P_t - (\mathcal{P}_{t/2}^* \partial_j A_{j,n+1}^*) P_t \right)$$

kills constants and satisfies $L^2 - L^2$ off-diagonal estimates (see, e.g., [AHLMcT]), so that by Lemma 4.4 one can bound $III'_{2,2}$ similarly to $III_{2,1}$. Finally,

$$\begin{aligned} (4.20) \quad III''_{2,2} &\lesssim \sum_{j=1}^n \left\| C(t \mathcal{P}_{t/2}^* \partial_j A_{j,n+1}^*) \right\|_{L^{\infty}} \times \\ &\quad \times \left\| \mathcal{A} \left(\left(C_1 t^3 L^2 e^{-\frac{3}{4}t^2 L} + C_2 t L e^{-\frac{3}{4}t^2 L} \right) f \right) \right\|_{L^p} \|N_*(P_t \partial_t v)\|_{L^{p'}}. \end{aligned}$$

The latter estimate follows from the duality of tent spaces T_2^p and $T_2^{p'}$ and the fact that $T_2^{\infty} \cdot T_{\infty}^{p'} \hookrightarrow T_2^{p'}$ (see [CMS] for the case $p' > 2$ and [HMM] or [CV], p. 313 for all $0 < p' < \infty$).

For each j , the estimate on the Carleson measure of $C(t\mathcal{P}_{t/2}^* \partial_j A_{j,n+1}^*)$ is controlled, since the Kato problem for L_{\parallel}^* is equivalent to the statement that

$$\|C(t\mathcal{P}_{t/2}^* \nabla_{\parallel} \cdot \vec{b})\|_{L^\infty(\mathbb{R}^n)} \leq C \|\vec{b}\|_{L^\infty(\mathbb{R}^n)},$$

for every $\vec{b} \in L^\infty(\mathbb{R}^n, \mathbb{C}^n)$ [AHLMT]. Hence,

$$(4.21) \quad III_{2,2}'' \lesssim \|\nabla_{\parallel} f\|_{L^p} \|N_*(\partial_t v)\|_{L^{p'}}.$$

Indeed, recall that P_t is a “nice” approximation of identity, e.g., the heat semigroup of the Laplacian and hence, one can drop it inside the non-tangential maximal function using the fact that $N_*(P_t(F(\cdot, t))) \leq M(N_* F)$ with M denoting the Hardy-Littlewood maximal function. \square

This finishes the proof of Theorem 3.1.

5. BOUNDEDNESS OF LAYER POTENTIALS AND ESTIMATES FOR SOLUTIONS IN \mathbb{R}_+^{n+1} .

Let us start by recalling a few known estimates on the layer potentials and related operators that will be used throughout this section.

Proposition 5.1. *Suppose that L is t -independent, that L and its adjoint L^* satisfy the De Giorgi-Nash-Moser bound (1.6). Then the single layer potential satisfies the following estimates in L^2 :*

$$(5.2) \quad \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |t \partial_t \nabla S_t^L f(x)|^2 \frac{dx dt}{|t|} \leq C \|f\|_{L^2(\mathbb{R}^n)}^2,$$

$$(5.3) \quad \sup_{t \neq 0} \left(\| (S_t^{L^*} \nabla) f \|_{L^2(\mathbb{R}^n)} + \|\nabla S_t^L f\|_{L^2(\mathbb{R}^n)} \right) \lesssim \|f\|_{L^2(\mathbb{R}^n)},$$

$$(5.4) \quad \|\tilde{N}(\nabla S_{\pm t}^L f)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}.$$

Moreover, there exists $\varepsilon > 0$ such that for all $1 < p < 2 + \varepsilon$ we have

$$(5.5) \quad \|\tilde{N}(\nabla S_{\pm t}^L f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

and for the dual exponent p' ,

$$(5.6) \quad \|N_*((S_{\pm t}^{L^*} \nabla) f)\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p'}(\mathbb{R}^n)},$$

$$(5.7) \quad \|\mathcal{A}(t \nabla S_{\pm t}^{L^*} f)\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p'}(\mathbb{R}^n)}.$$

Analogous bounds hold for L^* . The implicit constants in all inequalities and $\varepsilon > 0$ depend upon the standard constants only.

Here and below

$$(5.8) \quad (S_t D_j) f(x) := \int_{\mathbb{R}^n} \frac{\partial}{\partial y_j} E(x, t; y, 0) f(y) dy, \quad 1 \leq j \leq n,$$

$$(5.9) \quad (S_t D_{n+1}) f(x) := \int_{\mathbb{R}^n} \frac{\partial}{\partial s} E(x, t; y, s) \Big|_{s=0} f(y) dy,$$

and we set

(5.10)

$$(S_t \nabla) := \left((S_t D_1), (S_t D_2), \dots, (S_t D_{n+1}) \right), \text{ and } (S_t \nabla) \cdot \vec{f} := \sum_{j=1}^{n+1} (S_t D_j) f_j,$$

where \vec{f} takes values in \mathbb{C}^{n+1} . Similarly,

(5.11)

$$(S_t \nabla_{\parallel}) := \left((S_t D_1), (S_t D_2), \dots, (S_t D_n) \right), \text{ and } (S_t \nabla_{\parallel}) \cdot \vec{f} := \sum_{j=1}^n (S_t D_j) f_j,$$

where \vec{f} takes values in \mathbb{C}^n .

Note that for t -independent operators, we have by translation invariance in t that $(S_t D_{n+1}) = -\partial_t S_t$.

Remark 5.12. We remark that, considering only the tangential gradient ∇_{\parallel} in (5.6), the latter may be re-formulated as

$$(5.13) \quad S_{\pm t}^{L^*} : L_{-1}^{p'} \rightarrow T_{\infty}^{p'}(\mathbb{R}_+^{n+1}).$$

We further note that (5.5), (5.6), t -independence of L , and the De Giorgi-Nash-Moser bounds immediately imply that

$$(5.14) \quad \partial_t S_t^L : L^p(\mathbb{R}^n) \rightarrow T_{\infty}^p(\mathbb{R}_+^{n+1}), \quad 1 < p < \infty,$$

under the assumptions of Proposition 5.1.

Proof of Proposition 5.1. The square function bound (5.2) was proved in [R] (for an alternative proof, see [GH]). The fact that (5.2) implies (5.3) and (5.4) is basically a combination of results in [AAHK] and [AA]. See Proposition 1.19 in [HMM] for a detailed discussion and references. The fact that (5.3) for L and L^* implies (5.5) and (5.6) has been proved in [HMiMo, Theorem 1.1 and estimate (4.45)]. Finally, (5.7) can be found in [HMM]. \square

Proposition 5.15. *Suppose that L is t -independent, that L and its adjoint L^* satisfy the De Giorgi-Nash-Moser bound (1.6). Then*

$$(5.16) \quad \nabla L^{-1} \operatorname{div} : T_2^p \rightarrow \tilde{T}_{\infty}^p, \quad 1 < p < 2 + \varepsilon,$$

$$(5.17) \quad \nabla L^{-1} \frac{1}{t} : T_1^p \rightarrow \tilde{T}_{\infty}^p, \quad 1 < p < 2 + \varepsilon,$$

for some $\varepsilon > 0$, depending on the standard constants only. Here the operator $L^{-1} \frac{1}{t}$ is to be interpreted via

$$(5.18) \quad (L^{-1} \frac{1}{t} \Psi)(y, s) := \iint_{\mathbb{R}_+^{n+1}} E(y, s; x, t) \Psi(x, t) \frac{dx dt}{t}, \quad (y, s) \in \mathbb{R}_+^{n+1}.$$

Proof. The Proposition was proved in [HMM]. \square

Proposition 5.19. *Suppose that L is t -independent, that L and its adjoint L^* satisfy the De Giorgi-Nash-Moser bound (1.6). Then there exists $\varepsilon > 0$ such that for all $1 < p < 2 + \varepsilon$, we have*

$$(5.20) \quad \nabla D_{\pm t}^L : \dot{L}_1^p \rightarrow T_{\infty}^p(\mathbb{R}_+^{n+1}).$$

In particular,

$$(5.21) \quad \|\tilde{N}((\nabla D_{\pm t}^L f))\|_{L^p(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

Here $C > 0$ and $\varepsilon > 0$ depend upon the standard constants only. Analogous bounds hold for L .

Proof. By the usual density considerations, it is enough to demonstrate (5.21) for $f \in C_0^\infty(\mathbb{R}^n)$. To start, let us concentrate on the estimate for $\partial_t D_t^L$, that is,

$$(5.22) \quad \|\tilde{N}(\partial_t D_t^L f)\|_{L^p(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

To this end, we note that $L_{y,s}^* E^*(y, 0; x, t) = 0$ for all $(x, t) \in \mathbb{R}_+^{n+1}$, since $(y, 0)$ is on the boundary. Hence, one can formally write

$$(5.23) \quad \begin{aligned} \partial_t \int_{\mathbb{R}^n} \overline{\partial_{v_{A^*}, y} E^*(y, 0; x, t)} f(y) dy &= \int_{\mathbb{R}^n} \overline{\partial_s e_{n+1} A^*(y) \nabla_{y,s} E^*(y, s; x, t)} \Big|_{s=0} f(y) dy \\ &= - \int_{\mathbb{R}^n} \sum_{j=1}^n \overline{\partial_j e_j A^*(y) \nabla_{y,s} E^*(y, s; x, t)} \Big|_{s=0} f(y) dy \\ &= \int_{\mathbb{R}^n} \overline{L_{\parallel, y}^* E^*(y, 0; x, t)} f(y) dy - \int_{\mathbb{R}^n} \sum_{j=1}^n \overline{\partial_j A_{j,n+1}^*(y) \partial_s E^*(y, s; x, t)} \Big|_{s=0} f(y) dy \\ &= \int_{\mathbb{R}^n} \overline{A_{\parallel}^*(y) \nabla_y E^*(y, 0; x, t)} \nabla_y f(y) dy \\ &\quad + \int_{\mathbb{R}^n} \sum_{j=1}^n \overline{A_{j,n+1}^*(y) \partial_s E^*(y, s; x, t)} \Big|_{s=0} \partial_j f(y) dy. \end{aligned}$$

Of course, the formal computations in (5.23) should be interpreted in the weak sense, using, in particular, the weak definition of the normal derivative (2.10) (see the discussion in proof of Theorem 5.35 for more details).

The desired estimate on the second term on the right-hand side of (5.23) follows from (5.14). Passing to the first term, recall that the tent space \tilde{T}_∞^p can be realized as a space of the linear functionals on $\tilde{T}_1^{p'}$, where the latter is defined as a collection of $F \in \mathbf{M}(\mathbb{R}_+^{n+1})$ such that

$$\|F\|_{\tilde{T}_1^{p'}} := \|C_1(W_2 F)\|_{L^{p'}} < \infty,$$

where

$$W_2 F(x, t) = \left(\iint_{W(x,t)} |F(y, s)|^2 dy ds \right)^{\frac{1}{2}}, \quad (x, t) \in \mathbb{R}_+^{n+1}.$$

Indeed, it was proved in [HR], Theorem 3.2, that $\tilde{T}_\infty^p = \left(\tilde{T}_1^{p'} \right)^*$, $1 < p < \infty$. Pairing the first term on the right-hand side of (5.23) with Φ , one obtains

$$\begin{aligned}
(5.24) \quad & \iint_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}^n} \overline{A_{\parallel}^*(y) \nabla_y E^*(y, 0; x, t)} \nabla_y f(y) dy \overline{\Phi(x, t)} \frac{dx dt}{t} \\
&= \int_{\mathbb{R}^n} \overline{A_{\parallel}^*(y) \nabla_y T\Phi(y, 0)} \nabla_y f(y) dy,
\end{aligned}$$

where $T = (L^*)^{-1} \frac{1}{t}$, that is, as before,

$$(5.25) \quad T\Phi(y, s) = T_s\Phi(y) := \iint_{\mathbb{R}_+^{n+1}} E^*(y, s; x, t) \Phi(x, t) \frac{dx dt}{t}. \quad (y, s) \in \mathbb{R}_+^{n+1}.$$

The goal is to show that

$$(5.26) \quad \left| \int_{\mathbb{R}^n} \overline{A_{\parallel}^*(y) \nabla_y T\Phi(y, 0)} \nabla_y f(y) dy \right| \lesssim \|\Phi\|_{\tilde{T}_1^{p'}} \|\nabla_{\parallel} f\|_{L^p},$$

for any $\Phi \in \tilde{T}_1^{p'}$.

It is sufficient to verify (5.26) for Φ smooth and compactly supported in \mathbb{R}_+^{n+1} , as such functions are dense in $\tilde{T}_1^{p'}$. At this point, recall the estimate (4.11) or, more precisely, (4.12). One can carefully track the proof of (4.11)–(4.12) to see that all the computations are justified for $v := T\Phi$ with Φ smooth and compactly supported in \mathbb{R}_+^{n+1} . However, $v = T\Phi$ is not a solution in \mathbb{R}_+^{n+1} and hence, the term III_1 (that is, the last integral on the right-hand side of (4.12)) will not be annulated. Instead, we have $L^*v(x, t) = L^*T\Phi(x, t) = \Phi(x, t)/t$ for $(x, t) \in \mathbb{R}_+^{n+1}$.

All in all, it is enough to bound the right-hand side of (4.12) with $v = T\Phi$, $\Phi \in \tilde{T}_1^{p'}(\mathbb{R}_+^{n+1})$. Thus, one has to prove:

$$\|\mathcal{A}(t \nabla \partial_t T_t \Phi)\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|\Phi\|_{\tilde{T}_1^{p'}(\mathbb{R}_+^{n+1})}, \quad \|N_*(\partial_t T_t \Phi)\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|\Phi\|_{\tilde{T}_1^{p'}(\mathbb{R}_+^{n+1})},$$

or, equivalently, in the language of tent spaces,

$$(5.27) \quad \|t \nabla \partial_t T_t \Phi\|_{T_2^{p'}(\mathbb{R}_+^{n+1})} \lesssim \|\Phi\|_{\tilde{T}_1^{p'}(\mathbb{R}_+^{n+1})},$$

$$(5.28) \quad \|\partial_t T_t \Phi\|_{T_{\infty}^{p'}(\mathbb{R}_+^{n+1})} \lesssim \|\Phi\|_{\tilde{T}_1^{p'}(\mathbb{R}_+^{n+1})},$$

and to bound the last term on the right-hand side of (4.12) with \bar{f} in place of f . However,

$$(5.29) \quad \iint_{\mathbb{R}_+^{n+1}} \partial_t^2 \mathcal{P}_t \bar{f}(x) L^* T\Phi(x, t) t dt dx = \iint_{\mathbb{R}_+^{n+1}} \partial_t^2 \mathcal{P}_t \bar{f}(x) \Phi(x, t) t dt dx,$$

so that using once again duality relationship for tent spaces the desired bound on (5.29) reduces to

$$(5.30) \quad \|\tilde{N}(t \partial_t^2 \mathcal{P}_t \bar{f})\|_{L^p(\mathbb{R}^n)} \lesssim \|\overline{\nabla_{\parallel} f}\|_{L^p(\mathbb{R}^n)} = \|\nabla_{\parallel} f\|_{L^p(\mathbb{R}^n)}.$$

The estimate (5.30) follows from the bound $\tilde{N}(t \partial_t^2 \mathcal{P}_t f) \lesssim M(\nabla_{\parallel} f)$, while the latter can be proved essentially by the same argument as that in (4.6)–(4.9), using the off-diagonal decay estimates on $t^2 \partial_t^2 \mathcal{P}_t$ and the Poincaré inequality.

It remains to discuss (5.27) and (5.28). A direct computation shows that the adjoint of the operator $\partial_t T$, under the usual tent space pairing $\langle \Phi, \Psi \rangle = \iint_{\mathbb{R}_+^{n+1}} \Phi \Psi \frac{dx dt}{t}$, is $-\partial_s L^{-1} \frac{1}{t}$ (since by t -independence one can swap the derivatives in t and s on the

fundamental solution). Since $T_\infty^p = \left(T_1^{p'}\right)^*$, $1 < p < \infty$, (see, e.g., [HR], Theorem 3.2, or the duality argument in [CMS]), (5.28) follows from (5.17) in the range $1 < p < 2 + \varepsilon$.

Finally, the adjoint of the operator $t\nabla\partial_t T$, under the tent space pairing $\langle \Phi, \Psi \rangle = \iint_{\mathbb{R}_+^{n+1}} \Phi \Psi \frac{dxdt}{t}$, is $-\partial_s L^{-1} \operatorname{div}$, and $T_2^p = \left(T_2^{p'}\right)^*$, $1 < p < \infty$, (see [CMS]), so that (5.27) follows from (5.16), once again, in the range $1 < p < 2 + \varepsilon$, as desired.

This finishes the proof of (5.22).

We claim that the desired bound (5.21) follows from (5.22), or, to be precise, from the estimate

$$(5.31) \quad \|N_*(\partial_t D_t^L f)\|_{L^p(\mathbb{R}^n)} \leq C \|\nabla_\parallel f\|_{L^p(\mathbb{R}^n)}.$$

Since $\partial_t D_t^L f$ is a solution, (5.31) is equivalent to (5.22) by the De Giorgi-Nash-Moser estimates.

Indeed, using the argument in [KP], p. 494, we see that the left-hand side of (5.21) is controlled by the sum of $\|M(\nabla_\parallel D_t^L)|_{t=0} f)\|_{L^p(\mathbb{R}^n)}$ and $\|M(N_*(\partial_t D_t^L f))\|_{L^p(\mathbb{R}^n)}$, where M is the Hardy-Littlewood maximal function on \mathbb{R}^n . The first of these two terms is bounded by $\|\nabla_\parallel D_t^L|_{t=0} f\|_{L^p}$. The second one is bounded by the left-hand side of (5.31).

Thus, it remains to show that

$$(5.32) \quad \nabla_\parallel D_t^L|_{t=0} : \dot{L}_1^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n), \quad 1 < p < 2 + \varepsilon.$$

Let $f, g \in C_0^\infty(\mathbb{R}^n)$, (where g is vector-valued) and set $w(x, t) := S_{-t}^{L^*} \operatorname{div}_\parallel g(x)$, $x \in \mathbb{R}^n$, so that, by (5.6),

$$(5.33) \quad \|w(\cdot, 0)\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

Note that w is a solution (for L^*) in the lower half-space, and choose an appropriate solution v in the lower half-space such that $\partial_t v = w$. Dualizing, and using the equation, we then have

$$(5.34) \quad \begin{aligned} \int_{\mathbb{R}^n} \nabla_\parallel D_t|_{t=0} f(x) \overline{g(x)} dx &= \int_{\mathbb{R}^n} f(x) \overline{\partial_{v_{A^*}} w(x)} dx = \int_{\mathbb{R}^n} f(x) \overline{\partial_t \partial_{v_{A^*}} v(x)} dx \\ &= - \sum_{j=1}^n \int_{\mathbb{R}^n} \partial_j f(x) \overline{A_{j,n+1}^* \partial_t v(x)} dx - \int_{\mathbb{R}^n} \nabla_\parallel f \overline{A_\parallel^* \nabla_\parallel v} dx =: I + II. \end{aligned}$$

For term I , we just use (5.33), and for term II , we apply estimate (3.9) and then (5.6) and (5.7) (one can go over the argument of (3.9) and justify integration by parts and convergence of involved integrals even though the data of w on the boundary is not C_0^∞). \square

Theorem 5.35. *Let L be an elliptic operator with t -independent coefficients such that the solutions to $Lu = 0$ and $L^*u = 0$ in \mathbb{R}_\pm^{n+1} satisfy the De Giorgi-Nash-Moser estimates. Let u be the solution to the Dirichlet problem $Lu = 0$ in \mathbb{R}_+^{n+1} , $u|_{\partial\mathbb{R}_+^{n+1}} = f$,*

for some $f \in C_0^\infty(\mathbb{R}^n)$, in the sense of Lemma 2.4. Then for $1 < p < 2 + \varepsilon$

$$(5.36) \quad \|\tilde{N}(\nabla u)\|_{L^p(\mathbb{R}^n)} \leq C \left(\|\nabla f\|_{L^p(\mathbb{R}^n)} + \|\partial_{\nu_A} u\|_{L^p(\mathbb{R}^n)} \right),$$

with C and ε depending on the standard constants only.

Proof. By Green's formula, the solution to $Lu = 0$ in \mathbb{R}_+^{n+1} , $u|_{\partial\mathbb{R}_+^{n+1}} = f$, can be written as

$$(5.37) \quad u(x, t) = - \int_{\mathbb{R}^n} \overline{\partial_{\nu_A^*, y} E^*(y, 0; x, t)} f(y) dy + \int_{\mathbb{R}^n} E(x, t; y, 0) \partial_{\nu_A} u(y) dy,$$

or, in the language of layer potentials,

$$(5.38) \quad u = -D_t^L(f) + S_t^L(\partial_{\nu_A} u).$$

With (5.38) at hand, (5.36) follows directly from (5.21) and (5.5).

It remains to justify the use of the Green's formula in the present context.

To this end, fix $X_0 := (x_0, t_0) \in \mathbb{R}_+^{n+1}$, and let $\varepsilon > 0$, with $\varepsilon < t_0/8$. We define a nice approximate identity on \mathbb{R}^{n+1} in the standard way as follows. Let $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$, with $\Phi \geq 0$, $\text{supp}(\Phi) \subset B(0, 1)$, and $\iint_{\mathbb{R}^{n+1}} \Phi = 1$, and set $\Phi_\varepsilon(X) := \varepsilon^{-n-1} \Phi(X/\varepsilon)$. For $(y, s) \in \mathbb{R}^{n+1}$, we set

$$H_\varepsilon(y, s) := \left(\Phi_\varepsilon * E^*(y, s, \cdot, \cdot) \right)(x_0, t_0).$$

We observe that by definition of the fundamental solution E^* ,

$$H_\varepsilon = (L^*)^{-1} \Phi_\varepsilon^{X_0},$$

where $\Phi_\varepsilon^{X_0}(X) := \Phi_\varepsilon(X_0 - X) \in C_0^\infty(B(X_0, \varepsilon))$. In particular, then, we have that $H_\varepsilon \in \dot{W}^{1,2}(\mathbb{R}^{n+1})$.

Let F be a $C_0^\infty(\mathbb{R}^{n+1})$ extension of $f = u(\cdot, 0)$, and set

$$\tilde{u}(x, t) := \begin{cases} u(x, t), & t \geq 0 \\ F(x, t), & t \leq 0. \end{cases}$$

Since $\varepsilon \ll t_0$, and since $L^* H_\varepsilon = \Phi_\varepsilon^{X_0}$ in the weak sense in \mathbb{R}^{n+1} (and in particular, $L^* H_\varepsilon = 0$ in the lower half-space), we have

$$(5.39) \quad \begin{aligned} \Phi_\varepsilon * u(X_0) &= \iint_{\mathbb{R}^{n+1}} \tilde{u}(X) \Phi_\varepsilon^{X_0}(X) dX = \iint_{\mathbb{R}^{n+1}} \nabla \tilde{u} \cdot \overline{A^* \nabla H_\varepsilon} \\ &= \iint_{\mathbb{R}_+^{n+1}} A \nabla u \cdot \overline{\nabla H_\varepsilon} + \iint_{\mathbb{R}_-^{n+1}} \nabla F \cdot \overline{A^* \nabla H_\varepsilon} = \langle \partial_{\nu_A} u, \overline{h_\varepsilon} \rangle - \langle \overline{\partial_{\nu_A^*} H_\varepsilon}, f \rangle, \end{aligned}$$

where $h_\varepsilon := H_\varepsilon(y, 0)$, and where in the last step we have used Lemma 2.11. By definition of H_ε , (5.38) follows, letting $\varepsilon \rightarrow 0$, once we establish the convergence in ε for both terms on the right-hand side of (5.39).

The convergence of the first term is quite easy, as $h_\varepsilon \rightarrow E^*$ in $\dot{L}_{1/2}^2(\mathbb{R}^n)$.

Let us discuss the second one. In this regard, one has to show that for $\varepsilon \ll t_0$, the variational co-normal derivative $-\partial_{\nu_A^*} H_\varepsilon$, initially defined in the sense of

Lemma 2.11, belongs to $L^2(\mathbb{R}^n)$, and satisfies

$$(5.40) \quad -\partial_{\nu_A^*} H_\varepsilon(y, 0) = e_{n+1} \cdot A^* \left(\Phi_\varepsilon * \nabla_{y,s} E^*(y, s, \cdot, \cdot) \Big|_{s=0} \right) (X_0).$$

That the right hand side of (5.40) belongs to $L^2(\mathbb{R}^n)$ follows directly from [AAAHK, Lemma 2.8]. Thus we need only verify the identity (5.40). By Lemma 2.11, applied in the lower half-space, it is enough to verify that for any $F \in C_0^\infty(\mathbb{R}^{n+1})$, with $F(\cdot, 0) := f$, we have

$$(5.41) \quad \iint_{\mathbb{R}_-^{n+1}} \nabla F \cdot \overline{A^* \nabla H_\varepsilon} = \int_{\mathbb{R}^n} f(y) \overline{\vec{N} \cdot A^* \left(\Phi_\varepsilon * \nabla_{y,s} E^*(y, s, \cdot, \cdot) \Big|_{s=0} \right) (X_0)} dy.$$

To this end, let P_η be a nice approximate identity in \mathbb{R}^n , applied in the y variable. By the divergence theorem

$$\begin{aligned} & \int_{\mathbb{R}^n} f(y) \overline{\vec{N} \cdot P_\eta \left(A^* \left(\Phi_\varepsilon * \nabla_{y,s} E^*(y, s, \cdot, \cdot) \Big|_{s=0} \right) (X_0) \right)} dy \\ &= \iint_{\mathbb{R}_-^{n+1}} \operatorname{div} \left(F(y, s) P_\eta \left(A^* \left(\Phi_\varepsilon * \nabla_{y,s} E^*(y, s, \cdot, \cdot) \Big|_{s=0} \right) (X_0) \right) \right) dy ds \\ &= \iint_{\mathbb{R}_-^{n+1}} \nabla F(y, s) \cdot \overline{P_\eta (A^* \nabla H_\varepsilon(y, s))} dy ds, \end{aligned}$$

where in the last step we have used the definition of H_ε , and that it is a null solution of L^* in \mathbb{R}_-^{n+1} . We now obtain (5.41) by letting $\eta \rightarrow 0$.

This finishes the argument. \square

Corollary 5.42. *Assume that L is an elliptic operator with t -independent coefficients, and that L and L^* satisfy the De Giorgi-Nash-Moser condition. If for some $1 < p < 2 + \varepsilon$ the Dirichlet problem $(D_{p'})$ for L^* is solvable in \mathbb{R}_\pm^{n+1} and, in addition to (1.4), the solution satisfies the square function estimates (1.12), then the Regularity problem (R_p) for L is solvable in \mathbb{R}_\pm^{n+1} . That is, assertion (a) implies (b) in Theorem 1.11.*

Proof. The combination of Theorems 5.35 and 3.1 entails that every solution to $Lu = 0$ in \mathbb{R}_+^{n+1} , $u|_{\partial\mathbb{R}_+^{n+1}} = f \in C_0^\infty(\mathbb{R}^n)$ in the weak sense of Lemma 2.4 satisfies the estimate (1.5). With this at hand, the solvability for a general elliptic operator L , as defined before the statement of Theorem 1.11, follows immediately. \square

6. INVERTIBILITY OF LAYER POTENTIALS. LAYER POTENTIAL REPRESENTATIONS OF SOLUTIONS

Propositions 5.1 and Remark 5.12 ascertain that the layer potentials are always bounded for elliptic operators with t -independent coefficients that satisfy the De Giorgi-Nash-Moser bounds. We note that, furthermore, (5.5) and Lemma 2.2 imply that for $f \in L^p(\mathbb{R}^n)$, $1 < p < 2 + \varepsilon$, the single layer potential $S^L f$ has well-defined normal and tangential derivatives on the boundary in the sense of Lemma 2.2 (denoted below by $-\vec{e}_n \cdot A \nabla S^L f$ and $\nabla_\parallel S^L f$, respectively). Both $\vec{e}_n \cdot A \nabla S^L f$ and $\nabla_\parallel S^L f$ belong to L^p with the appropriate estimates. In particular,

$$(6.1) \quad S^L : L^p(\mathbb{R}^n) \rightarrow \dot{L}_1^p(\mathbb{R}^n),$$

is a bounded operator for all $1 < p < 2 + \varepsilon$. When necessary, we shall use the superscripts $+$ and $-$ to underline the limit taken from the upper and the lower half space, respectively.

Recall the following jump relations:

$$(6.2) \quad \vec{e}_n \cdot A(\nabla S^L)^+ g - \vec{e}_n \cdot A(\nabla S^L)^- g = -g,$$

$$(6.3) \quad (\nabla_{\parallel} S^L)^+ g - (\nabla_{\parallel} S^L)^- g = 0,$$

$$(6.4) \quad (D^L)^+ f - (D^L)^- f = -f.$$

The jump relations (6.2), (6.3) can be found, e.g., in the proof of [AAAHK, Lemma 4.18] for $g \in L^2(\mathbb{R}^n)$, and then extended to any $g \in L^p(\mathbb{R}^n)$, $1 < p < 2 + \varepsilon$, using (5.5) and Lemma 2.2. In particular $(S^L)^+ = (S^L)^-$ regarded as operators $L^p(\mathbb{R}^n) \rightarrow \dot{L}_1^p(\mathbb{R}^n)$, $1 < p < 2 + \varepsilon$. Similarly, (6.4) is established for $f \in C_0^\infty(\mathbb{R}^n)$ in [AAAHK, Lemma 4.18] and can be extended by continuity to $f \in \dot{L}_1^p(\mathbb{R}^n)$ for all p such that (5.21) holds.

Proposition 6.5. *Assume that L is an elliptic operator with t -independent coefficients, and that L and L^* satisfy the De Giorgi-Nash-Moser condition. Assume further that for some $1 < p' < \infty$ the Dirichlet problem $(D_{p'})$ for the operator L^* is solvable with the square function bounds, that is, the solution u satisfies both (1.4) and the square function estimate (1.12). Then the operator $(S^L)^\pm : L^p(\mathbb{R}^n) \rightarrow \dot{L}_1^p(\mathbb{R}^n)$ is invertible, $\frac{1}{p} + \frac{1}{p'} = 1$. An analogous statement holds for L^* .*

Proof. First of, the boundedness of $(S^L)^\pm$ follows from the discussion of (6.1) above. The bounds from below can be established as follows. By (6.2), (6.3) and Lemma 2.2, for any $g \in C_0^\infty(\mathbb{R}^n)$ we have

$$(6.6) \quad \|g\|_{L^p(\mathbb{R}^n)} = \|\vec{e}_n \cdot A(\nabla S^L)^+ g - \vec{e}_n \cdot A(\nabla S^L)^- g\|_{L^p(\mathbb{R}^n)} \leq C \|(\nabla_{\parallel} S^L g)^\pm\|_{L^p(\mathbb{R}^n)}.$$

To justify this, we point out that whenever $g \in C_0^\infty(\mathbb{R}^n)$, the functions $(S^L)^\pm g$ belong to the space $L_1^2(\mathbb{R}^n) \cap L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ from Lemma 2.4 and to $\dot{L}_1^p(\mathbb{R}^n)$. This can be seen from (6.1) and its dual. Hence, $(S^L)^\pm g$ can be approximated by $f_k \in C_0^\infty(\mathbb{R}^n)$ in $L_1^2(\mathbb{R}^n) \cap L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ and in $\dot{L}_1^p(\mathbb{R}^n)$. Using Lemma 2.5 in [AAAHK], one can also show that the solution $S_t g$ belongs to $\tilde{W}^{1,2}(\mathbb{R}_+^{n+1})$. Now, for each such f_k there exists a weak solution of the Dirichlet problem by Lemma 2.4 (call it u_k) and, due to (2.5), these solutions converge to $S_t g$ in $\tilde{W}^{1,2}(\mathbb{R}^n)$ norm. Moreover, u_k satisfy Theorem 3.1. Hence, the corresponding weak conormal derivatives $\partial_{\nu_A} u_k$ form a Cauchy sequence in L^p and thus, converge to some L^p function, h , in L^p norm. On the other hand, as mentioned above, u_k converge to $S_t g$ in $\tilde{W}^{1,2}(\mathbb{R}^n)$ norm and hence, $\partial_{\nu_A} u_k$ converge to the conormal derivative of $S_t g$ in the sense of distributions. Thus, the conormal derivative of $S_t g$ can be identified with h and its L^p norm is bounded by the L^p norm of the tangential derivative, as desired.

Next, we note that (6.6) for all $g \in C_0^\infty(\mathbb{R}^n)$ implies (6.6) for all $g \in L^p(\mathbb{R}^n)$. Indeed, every $g \in L^p(\mathbb{R}^n)$ can be approximated in L^p by C_0^∞ functions g_k , and by (6.1) $(\nabla_{\parallel} S^L g)^\pm$ is well-defined, belongs to L^p , and the sequence $(\nabla_{\parallel} S^L g_k)^\pm$ converges in L^p to $(\nabla_{\parallel} S^L g)^\pm$. Thus, (6.6) holds for all $g \in L^p(\mathbb{R}^n)$ as well.

Let us show that $(S^L)^\pm$ is surjective². Choose some $f \in C_0^\infty(\mathbb{R}^n)$. Let u^\pm be the solutions to the Dirichlet problem in the weak sense of Lemma 2.4 with boundary data f in \mathbb{R}_\pm^{n+1} . By Lemma 2.2, the distributions $g^\pm = \vec{e}_n \cdot A \nabla u^\pm \Big|_{\partial \mathbb{R}_\pm^{n+1}}$ lie in $L^p(\mathbb{R}^n)$.

We shall show that

$$(6.7) \quad (S^L)^\pm(g_- - g_+) = f \quad \text{in} \quad \dot{L}_1^p(\mathbb{R}^n).$$

To this end, observe first that u^\pm are also the solutions to the regularity problem (R_p) (see, e.g., the proof of Corollary 5.42) and moreover, according to the proof of Theorem 5.35, we have

$$u^\pm = -(D_t^L)^\pm(f) + (S_t^L)^\pm(g^\pm).$$

Using jump relations (6.2)–(6.4), we deduce that

$$u^+ \Big|_{\mathbb{R}^n} - u^- \Big|_{\mathbb{R}^n} = f + (S^L)^\pm(g_+ - g_-).$$

Restriction to the boundary is interpreted here in the sense of the non tangential limit of the gradient, and the equality holds in $\dot{L}_1^p(\mathbb{R}^n)$. We used, in particular, the fact that $(S^L)^+ = (S^L)^-$ regarded as operators $L^p(\mathbb{R}^n) \rightarrow \dot{L}_1^p(\mathbb{R}^n)$, alluded to above. On the other hand, by definition of u as a solution to (R_p) with data f , we have $u^+ \Big|_{\mathbb{R}^n} - u^- \Big|_{\mathbb{R}^n} = 0$, again in the sense of the non-tangential limit of the gradient and equality in $\dot{L}_1^p(\mathbb{R}^n)$. The combination of these two facts immediately yields (6.7) for $f \in C_0^\infty(\mathbb{R}^n)$. Now, for general $f \in \dot{L}_1^p(\mathbb{R}^n)$ we can use the limiting procedure, (6.7) for C_0^∞ functions and (6.6) to find $g \in L^p(\mathbb{R}^n)$ such that $(S^L)^\pm g = f$. \square

Corollary 6.8. *Assume that L is an elliptic operator with t -independent coefficients, and that L and L^* satisfy the De Giorgi-Nash-Moser condition.*

If (R_p) is well-posed in \mathbb{R}_\pm^{n+1} for some $1 < p < 2 + \varepsilon$ then the solution can be represented by means of layer potentials, that is, assertion (b) implies (c) in Theorem 1.11. An analogous statement holds for L^ .*

Proof. The corollary is a combination of the boundedness results in Proposition 5.1 and invertibility of layer potentials following from Proposition 6.5. \square

Corollary 6.9. *Assume that L is an elliptic operator with t -independent coefficients, and that L and L^* satisfy the De Giorgi-Nash-Moser condition.*

If (R_p) is well-posed in \mathbb{R}_\pm^{n+1} for L for some $1 < p < 2 + \varepsilon$ with the solution represented by means of layer potentials then $(D_{p'})$ is well-posed in \mathbb{R}_\pm^{n+1} for L^ , with the solution represented by means of layer potentials, that is, assertion (c) implies (d) in Theorem 1.11. An analogous statement holds for L^* .*

Proof. By Proposition 6.5 and preceding discussion, the operator $(S^L)^\pm : L^p(\mathbb{R}^n) \rightarrow \dot{L}_1^p(\mathbb{R}^n)$ is bounded and invertible. It follows that the operator

$$(S^{L^*})^\pm : L_{-1}^{p'}(\mathbb{R}^n) \rightarrow L^{p'}(\mathbb{R}^n)$$

²We note that the surjectivity argument presented here was inspired by the surjectivity result for layer potentials in [BM]

is bounded and invertible. Taking in account (5.6), the solvability of the Dirichlet problem via the representation (1.15) follows immediately once we observe that $h \in L_{-1}^{p'}$ if and only if there exists $\vec{H} \in L^{p'}$ such that $\operatorname{div}_{\parallel} \vec{H} = h$, with the accompanying norm equivalence. \square

7. WELL-POSEDNESS AND UNIQUENESS

Sections 3–6 establish equivalence of solvability of the Regularity and Dirichlet boundary value problems, in the context of Theorem 1.11. However, even having solvability at hand, well-posedness turns out to be a delicate issue. Basically, under current assumptions on operator, the notion of a solution as a $W_{loc}^{1,2}$ function satisfying appropriate non-tangential maximal function estimates does not seem to be sufficient to justify, e.g., the Green's formula. Thus, one has to appeal to a slightly weaker notion of uniqueness and, respectively, well-posedness of $(D_{p'})$, (R_p) in the current context.

Let us first note that for every $f \in C_0^\infty(\mathbb{R}^n)$ the solutions to $(D_{p'})$, (R_p) obtained in Theorem 1.11 are the weak solutions in the sense of Lemma 2.4. And, in the class of weak solutions, they are unique (again, by Lemma 2.4). Clearly, it is sufficient to justify this for layer potential representations (1.14), (1.15). To this end, observe that for $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\nabla S_t^L (S_0^L)^{-1} f \in \bigcap_{1 < p < 2+\varepsilon} \tilde{T}_\infty^p$$

by Theorem 1.11. Thus, in particular, $\nabla S_t^L (S_0^L)^{-1} f \in \tilde{T}_{\infty^{n+1}}^{\frac{2n}{n+1}}$ and hence, by Lemma 2.2 in [HMiMo], we have $\nabla S_t^L (S_0^L)^{-1} f \in L^2(\mathbb{R}_+^{n+1})$. Similarly,

$$S_t^L (S_0^L)^{-1} f \in \bigcap_{1 < p < 2+\varepsilon} T_\infty^{p'} \implies S_t^L (S_0^L)^{-1} f \in \bigcap_{1 < p < 2+\varepsilon} L^{\frac{p'(n+1)}{n}}(\mathbb{R}_+^{n+1})$$

implies that the $1/(1 + |X|^2)$ -weighted L^2 norm of $S_t^L (S_0^L)^{-1} f$ is finite. Altogether then, $S_t^L (S_0^L)^{-1} f \in \tilde{W}^{1,2}(\mathbb{R}_+^{n+1})$.

Having this information at hand, we can define the solution to $(D_{p'})$ (resp., (R_p)) with arbitrary data $f \in L^{p'}$ (resp., $f \in \dot{L}_1^{p'}$) as a limit. That is, one takes for any $f \in L^{p'}$ a sequence $\{f_k\}_k$ of C_0^∞ functions approximating f in $L^{p'}$ norm. Corresponding to each f_k , one can find a unique weak solution $u_k \in \tilde{W}^{1,2}(\mathbb{R}_+^{n+1})$ with the data f_k on the boundary. Then, if $(D_{p'})$ is solvable, u_k form a Cauchy sequence in $T_\infty^{p'}$. The limit of this sequence in $T_\infty^{p'}$ is the solution u with data f . By definition, u satisfies the desired estimates, and the standard argument shows that two different sequences f_k converging to f yield the same u . One moreover can easily verify that the solution obtained as such a limit also satisfies the layer potential representation (1.15) as $S_t^L (S_0^L)^{-1} f_k$ converge to $S_t^L (S_0^L)^{-1} f$ in the norm of $T_\infty^{p'}$.

Recall that the solutions in Theorem 1.11 were already defined by this limiting procedure (see the Remark after Lemma 2.2). The difference is that now we specify u_k for every f_k in a *unique* fashion. And hence, the solution u defined as a limit of such u_k 's is unique and the problem is well-posed. The same argument applies to the Regularity problem. All in all, both in the Dirichlet and Regularity case, one

has uniqueness of solutions (in the class of the limits of the weak solutions in the appropriate tent space) and thus, one obtains well-posedness in this sense.

In fact, we shall prove a stronger result. Recall that $(D_{p'})$ (resp., (R_p)) is *well-posed with the Green formula representation* if for every $f \in L^{p'}$ (resp., $f \in \dot{L}_1^p$) there exists a unique solution u satisfying (1.4) (resp., (1.5)) such that the Green formula (5.37) holds for u . Let us single out the following result.

Lemma 7.1. *Under the conditions of Theorem 1.11 the corresponding boundary problems are well-posed with the Green formula representations. In particular, uniqueness holds in the following sense. Take the operator L falling under the scope of Theorem 1.11 and, respectively, $1 < p < \infty$ such that the statements (a) – (d) are valid. Then whenever $u \in W_{loc}^{1,2}(\mathbb{R}_+^{n+1})$ is such that $N_*u \in L^{p'}(\mathbb{R}^n)$, $u|_{\partial\Omega} = 0$ and the Green formula (5.37) holds, we have $u \equiv 0$. Similarly, whenever $u \in W_{loc}^{1,2}(\mathbb{R}_+^{n+1})$ is such that $\tilde{N}(\nabla u) \in L^p(\mathbb{R}^n)$, $u|_{\partial\Omega} = 0$ and the Green formula (5.37) holds, we have $u \equiv 0$ ³.*

Proof. Indeed, first of all, as we discussed above, for every C_0^∞ data the solutions provided by Theorem 1.11 are the weak solutions. The proof of Theorem 5.35 demonstrated, in particular, that for every weak solution the Green formula representation (5.37)–(5.38) holds. Using mapping properties of the layer potentials (5.21), (5.5), and (2.3), one can show now that any solution to (R_p) defined by the limiting procedure as above also satisfies (5.37)–(5.38).

Similarly, using (5.6), we see that any solution to $(D_{p'})$ defined by the limiting procedure as above satisfies (5.37)–(5.38) provided that $\partial_{v_A} u_k$ converge to $\partial_{v_A} u$ in $L_{-1}^{p'}(\mathbb{R}^n)$. However, $u_k = S_t^L(S_0^L)^{-1}f_k$ for all $k \in \mathbb{N}$ and, as we discussed above, u defined by the limiting procedure necessarily satisfies $u = S_t^L(S_0^L)^{-1}f$. Thus, one just has to show that $\partial_{v_A} S_t^L|_{t=0}$ is a bounded operator in $L_{-1}^{p'}$. This, however, is exactly the dual statement to (5.32).

It remains to show that solutions satisfying the Green formula (5.37)–(5.38) are unique. To this end, assume, for instance, that u solves (R_p) with $f \equiv 0$ and satisfies (5.38). Then $u = S_t^L(\partial_{v_A} u)$. Taking the non-tangential limit on the boundary (by Lemma 2.2) and using invertibility of the single layer potential established in Proposition 6.5, we deduce that $\partial_{v_A} u \equiv 0$. Hence, u given by (5.37)–(5.38) is identically zero, as desired.

Turning to the Dirichlet problem, we first have to explain what is meant by a solution with the Green formula representation, or more precisely, how the normal derivative of an arbitrary $u \in W_{loc}^{1,2}(\mathbb{R}_+^{n+1})$ with $Lu = 0$ and $N_*u \in L^{p'}(\mathbb{R}^n)$ is defined.

Choose $1 < p' < \infty$ such that $(D_{p'})$ is solvable with the square function bounds, and assume that $u \in W_{loc}^{1,2}(\mathbb{R}_+^{n+1})$ with $Lu = 0$ and $N_*u \in L^{p'}(\mathbb{R}^n)$ is a solution with data $u|_{\mathbb{R}^n} = f \in L^{p'}$.

³The data of the Regularity problem are defined modulo constants (as elements of \dot{L}_1^p), and as usual, the uniqueness is understood in the sense of working with a particular realization of the data equal to zero a.e.

Let us define a tempered distribution $\partial_{v_A} u$ as

$$(7.2) \quad \langle \partial_{v_A} u, \varphi \rangle := \langle f, \partial_{v_A^*} \Phi \rangle, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n),$$

where Φ is the unique weak solution to $L^* \Phi = 0$, $\Phi|_{\mathbb{R}^n} = \varphi$ in the sense of Lemma 2.4. Note that the definition coincides with the one used before when, e.g., u is the unique weak solution to $Lu = 0$, $u|_{\mathbb{R}^n} = f$ in the sense of Lemma 2.4.

Let us show that $\partial_{v_A} u$, defined by (7.2), belongs to $\dot{L}_{-1}^{p'}(\mathbb{R}^n) = (\dot{L}_1^p(\mathbb{R}^n))^*$. Indeed, we have taken $1 < p' < \infty$ such that $(D_{p'})$ is solvable. Thus, by Theorem 1.11, the Regularity problem (R_p) is solvable, and moreover, as we discussed above, for every $\varphi \in C_0^\infty$ the solution is given by the unique weak solution in the sense of Lemma 2.4 and thus, Φ in (7.2) satisfies the estimate

$$\|\partial_{v_A^*} \Phi\|_{L^p(\mathbb{R}^n)} \lesssim \|\varphi\|_{\dot{L}_1^p(\mathbb{R}^n)},$$

by Theorem 3.1. It follows that

$$|\langle \partial_{v_A} u, \varphi \rangle| \leq C \|f\|_{L^{p'}(\mathbb{R}^n)} \|\varphi\|_{\dot{L}_1^p(\mathbb{R}^n)},$$

and hence, $\partial_{v_A} u \in \dot{L}_{-1}^{p'}(\mathbb{R}^n)$. The remainder of the uniqueness argument follows the same lines as that for the Regularity problem. \square

Remark. Concluding this Section, we note that whenever L and L^* satisfy the De Giorgi-Nash-Moser bounds in \mathbb{R}_\pm^{n+1} and at the boundary, the Regularity and the Dirichlet problems are well-posed in the usual strong sense (without explicitly requiring validity of the Green formula). That is, whenever $u \in W_{loc}^{1,2}(\mathbb{R}_+^{n+1})$ is such that $N_* u \in L^{p'}(\mathbb{R}^n)$, $u|_{\partial\Omega} = 0$, we have $u \equiv 0$. And similarly, whenever $u \in W_{loc}^{1,2}(\mathbb{R}_+^{n+1})$ is such that $\tilde{N}(\nabla u) \in L^p(\mathbb{R}^n)$, $u|_{\partial\Omega} = 0$, we have $u \equiv 0$. This is the case, e.g., when the elliptic operator has real coefficients.

8. PROOFS OF COROLLARIES 1.16–1.19

We conclude the paper with the Proofs of Corollaries 1.16–1.19.

Proof of Corollary 1.16. The De Giorgi-Nash-Moser bounds are stable under L^∞ perturbation of the coefficients. Thus, if L_0 falls under the scope of Theorem 1.11, so that L_0 and L_0^* both have the De Giorgi-Nash-Moser property, then so do L and L^* . Thus, the conditions (a) – (d) of Theorem 1.11 are equivalent for the operator L too, and it is sufficient to prove one of them. It is easier to access (c) or (d). Let us focus on (c).

The fact that $f \mapsto \tilde{N}(\nabla S_t^L f)$ is bounded in $L^p(\mathbb{R}^n)$, $1 < p < 2 + \varepsilon$, for any t -independent operator L (with some ε depending on the ellipticity constant) is known – see the discussion of (5.5). Respectively,

$$S_0^L : L^p(\mathbb{R}^n) \rightarrow \dot{L}_1^p(\mathbb{R}^n),$$

is bounded by Lemma 2.2. The operator norm in both cases depends on standard constants only.

As far as invertibility is concerned, we have by analytic perturbation theory (see the Appendix for details)

$$(8.1) \quad \|\nabla_{\parallel} S_0^L - \nabla_{\parallel} S_0^{L_0}\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \leq C \|A_0 - A\|_{L^\infty(\mathbb{R}^n)},$$

and hence, invertibility of S_0^L follows from that of $S_0^{L_0}$ via the Neumann series. \square

Proof of Corollary 1.17. This is essentially the main result of [HMM]. Let us provide some explanations.

It is clear that it is sufficient to prove (c) and (d) for L , together with the square function bound (1.12) for the solution given by (1.15). Then (a) and (b) would automatically follow. We note that equivalence of (a) – (d) for L in this case does not follow from Theorem 1.11, as the coefficients of L are not t -independent.

The fact (c) and (d) for L_0 imply (c) and (d) for L under the conditions of the Corollary is one the main theorems in [HMM]. \square

Proof of Corollary 1.19. Recall that the De Giorgi-Nash-Moser bounds for solutions inside the domain and at the boundary are always valid for the operators with real coefficients.

The validity of (a) for an elliptic operator with real, t -independent, possibly non-symmetric, coefficients is the main result of [HKMP]. Then the validity of all ascertions (b) – (d) of Theorem 1.11 for such an operator follows from Theorem 1.11, and the claimed perturbation results follow from Corollaries 1.16 and 1.17. \square

9. APPENDIX: ANALYTICITY OF ∇S_t

In this section we present a discussion of (8.1). It is essentially treated within the framework of the analytic perturbation theory in [Ka] (and was already used, e.g., in [AAAHK], [B]). However, a detailed argument for the particular case at hand does not seem to be available in the literature, and for completeness, we provide it here.

Let A_0 be elliptic, $(n+1) \times (n+1)$, t -independent, and bounded measurable, and let $A_z := A_0 + zM$, where M is $(n+1) \times (n+1)$, t -independent, and bounded measurable, with $\|M\|_{L^\infty(\mathbb{R}^n)} \leq 1$. Set $L_0 := -\operatorname{div} A_0 \nabla$, and $L_z := -\operatorname{div} A_z \nabla$, and suppose that null solutions of L_0 satisfy the De Giorgi-Nash-Moser bounds. For $|z|$ small enough, say $|z| < \varepsilon_0$, we have that L_z is also elliptic, and satisfies the De Giorgi-Nash-Moser bounds. By ellipticity, for $|z| < \varepsilon_0$,

$$\nabla L_z^{-1} \operatorname{div} : L^2(\mathbb{R}^{n+1}) \rightarrow L^2(\mathbb{R}^{n+1}),$$

or equivalently,

$$L_z^{-1} : \dot{W}^{-1,2}(\mathbb{R}^{n+1}) \rightarrow \dot{W}^{1,2}(\mathbb{R}^{n+1}).$$

Moreover, we have the Taylor expansion

$$\nabla L_z^{-1} \operatorname{div} = \nabla L_0^{-1} \operatorname{div} \sum_{k=0}^{\infty} (zM \nabla L_0^{-1} \operatorname{div})^k,$$

which is convergent, as a mapping from $\dot{W}^{-1,2}(\mathbb{R}^{n+1})$ to $\dot{W}^{1,2}(\mathbb{R}^{n+1})$, if ε_0 is small enough. Therefore, the mapping $z \rightarrow L_z^{-1}$, taking values in the space of bounded

operators from $\dot{W}^{-1,2}(\mathbb{R}^{n+1})$ to $\dot{W}^{1,2}(\mathbb{R}^{n+1})$, is analytic in a neighborhood of $z = 0$. The same is true for L_z^* , so by trace theory, we then have that $z \rightarrow \text{Tr} \circ (L_z^*)^{-1}$ is an analytic mapping, taking values in the space of bounded operators from $\dot{W}^{-1,2}(\mathbb{R}^{n+1})$ to $\dot{H}^{1/2}(\mathbb{R}^n)$. Here Tr denotes the trace operator, on $\mathbb{R}^n \times \{0\} = \partial(\mathbb{R}_+^{n+1})$. We then define the single layer potential for L_z , denoted by S^{L_z} , as the adjoint of the operator $\text{Tr} \circ (L_z^*)^{-1}$, so that $z \rightarrow S^{L_z}$ is an analytic mapping taking values in the space of bounded operators from $\dot{H}^{-1/2}(\mathbb{R}^n)$ to $\dot{W}^{1,2}(\mathbb{R}^{n+1})$. By trace theory again, we have that $z \rightarrow S_t^{L_z}$ is an analytic mapping taking values in the space of bounded operators from $\dot{H}^{-1/2}(\mathbb{R}^n)$ to $\dot{H}^{1,2}(\mathbb{R}^n)$, where $S_t^{L_z}$ denotes the restriction of S^{L_z} to the hyperplane $x_{n+1} = t$; i.e., $z \rightarrow \nabla_{\parallel} S_t^{L_z}$ is an analytic mapping taking values in the space of bounded operators from $\dot{H}^{-1/2}(\mathbb{R}^n)$ to $\dot{H}^{-1,2}(\mathbb{R}^n)$. Thus,

$$z \rightarrow \langle \nabla_{\parallel} S_t^{L_z} f, g \rangle$$

is an analytic function, for every $f \in \dot{H}^{-1/2}(\mathbb{R}^n)$, and every $g \in \dot{H}^{1/2}(\mathbb{R}^n)$, in particular, for every pair $f, g \in C_0^\infty(\mathbb{R}^n)$. By [HMiMo, Theorem 1.1], we have that

$$\sup_{|z| < \varepsilon_0} \sup_t \|\nabla_{\parallel} S_t^{L_z}\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \leq C_p, \quad 1 < p < 2 + \varepsilon,$$

with C_p depending only on p , ellipticity, dimension, and the the De Giorgi-Nash-Moser constants. Therefore by [Ka, p. 365], since C_0^∞ is dense in L^p and $L^{p'}$, we have that for each fixed t , $z \rightarrow \nabla_{\parallel} S_t^{L_z}$ is an analytic mapping, taking values in the space of bounded operators on $L^p(\mathbb{R}^n)$, in the disk $|z| < \varepsilon_0$. This means that

$$\sup_t \left\| \frac{d}{dz} \nabla_{\parallel} S_t^{L_z} \right\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \leq C_p,$$

so that

$$(9.1) \quad \sup_t \|\nabla_{\parallel} S_t^{L_z} - \nabla_{\parallel} S_t^{L_0}\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \leq C_p |z|.$$

Now, given A_0 as above, take $M := (A_1 - A_0) / \|A_1 - A_0\|_{L^\infty(\mathbb{R}^n)}$, so that $A_1 = A_0 + z_1 M$, with $z_1 = \|A_1 - A_0\|_{L^\infty(\mathbb{R}^n)}$. If $\|A_1 - A_0\|_{L^\infty(\mathbb{R}^n)} < \varepsilon_0$, by (9.1), we have that

$$\sup_t \|\nabla_{\parallel} S_t^{L_1} - \nabla_{\parallel} S_t^{L_0}\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \leq C_p \|A_1 - A_0\|_{L^\infty(\mathbb{R}^n)},$$

as desired.

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STEVE HOFMANN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA

E-mail address: hofmanns@missouri.edu

CARLOS KENIG, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL, 60637 USA

E-mail address: cek@math.chicago.edu

SVITLANA MAYBORODA, SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN, 55455 USA

E-mail address: svitlana@math.umn.edu

JILL PIPHER, DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI, USA

E-mail address: jpipher@math.brown.edu